# The Discrete Steklov-Poincaré Operator using dual Polynomials

## Multilevel and Multigrid Methods

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# Constrained minimization

Given a bounded domain $\Omega$ with a Lipschitz continuous boundary $\partial\Omega$. Find the vector field $\boldsymbol{u}$ with minimal $L^2$-norm subject to the constraint $\nabla \cdot \boldsymbol{u} = f$ for a given $f \in L^2(\Omega)$.

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Characterization of the affine subspace $\nabla \cdot \boldsymbol{u} = f$, generally not easy therefore, constraint included by means of Lagrange multipliers

**Mixed formulation**

For $\boldsymbol{v} \in H(div;\Omega)$ and $q \in L^2(\Omega)$ consider the functional

$$\mathcal{L}(\boldsymbol{v}, q) = \|\boldsymbol{v}\|^2_{L^2(\Omega)} + \int_\Omega q \, (\nabla \cdot \boldsymbol{v} - f) \, \mathrm{d}\Omega .$$

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The optimality conditions for this functional are given by

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$$\begin{cases} (\boldsymbol{v},\boldsymbol{u})_{L^2(\Omega)} + (\nabla\cdot\boldsymbol{v},p)_{L^2(\Omega)} = 0 & \forall \boldsymbol{v} \in H(div;\Omega) \\ (q,\nabla\cdot\boldsymbol{u})_{L^2(\Omega)} = (q,f)_{L^2(\Omega)} & \forall q \in L^2(\Omega) \end{cases}$$

This problem is well-posed. Note that here we solve the Poisson problem for $p$ with right hand side functions $f$ and $p = 0$ along the boundary $\partial\Omega$.

For non-homogeneous Dirichlet boundary conditions we take $p_b \in H^{\frac{1}{2}}(\partial\Omega)$ and modify the mixed formulation to

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yields a unique solution $\boldsymbol{u}$ and therefore a unique $\gamma(\boldsymbol{u}) = \boldsymbol{u} \cdot \boldsymbol{n} \in H^{-\frac{1}{2}}(\partial\Omega)$.

So we can define the map $\mathrm{S} \; : \; H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$ given by $S(p_b) = \boldsymbol{u} \cdot \boldsymbol{n}$.

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The Steklov-Poincaré operator is:

1. linear;
2. The operator is not surjective, because

$$\int_{\partial\Omega} \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma = \int_{\Omega} f\,\mathrm{d}\Omega$$

3. The operator is not injective. For $f \equiv 0$, $S(p) = 0$, when $p = const$.

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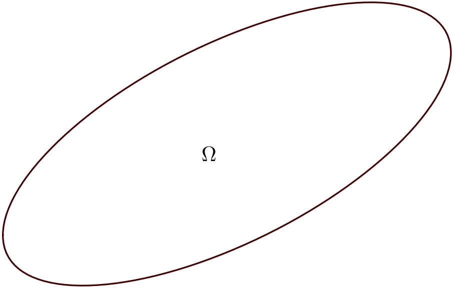
# Domain Decomposition I




The mixed formulation can be solved on a single domain.

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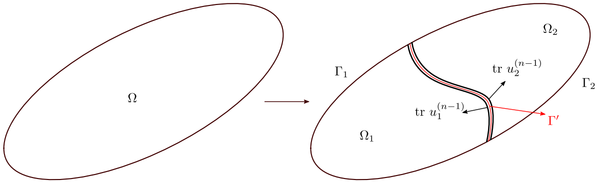


Domain decomposition

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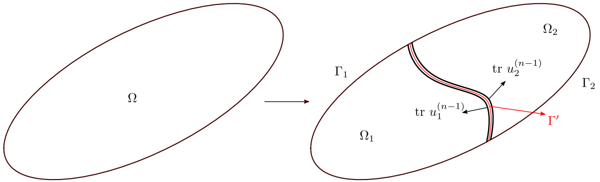


Domain decomposition

Advantage: Problems on sub-domains are smaller and can be solved in parallel

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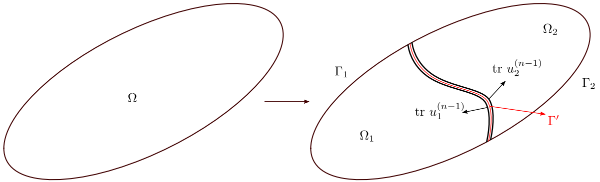


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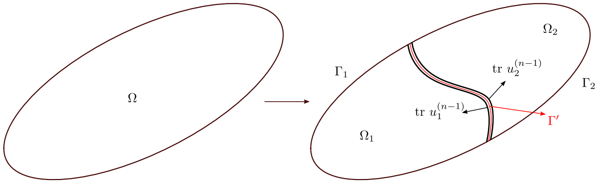


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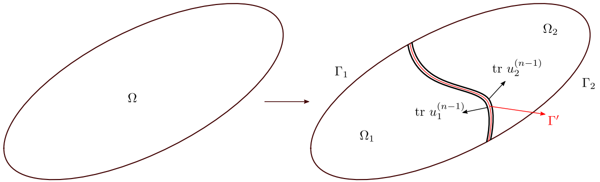


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This system is symmetric and indefinite.

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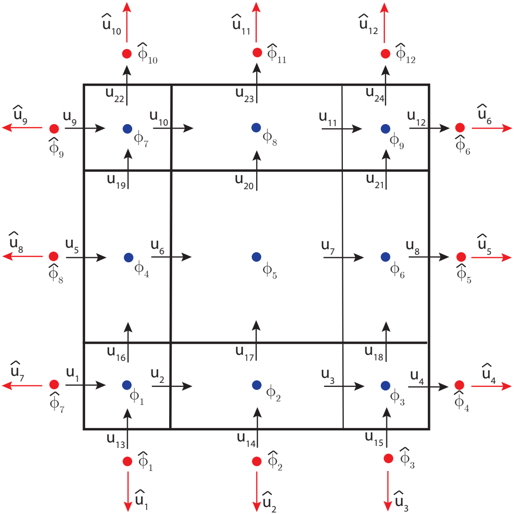
We can eliminate the internal unknowns within the elements and set up an equations for the interface unknowns only. This equation can be directly expressed in terms of the Steklov-Poincaré operators in both sub-domains.

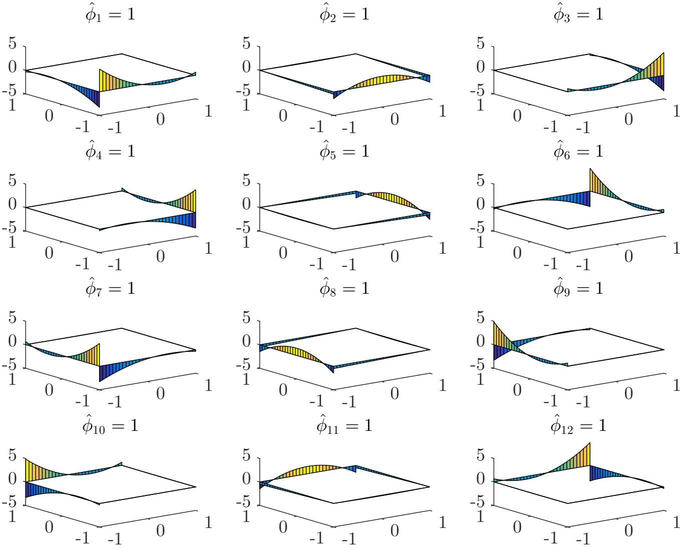
### Interface equation

$$\langle S_1(\bar{p}), \bar{q} \rangle + \langle S_2(\bar{p}), \bar{q} \rangle = 0 \qquad \forall \bar{q} \in H^{\frac{1}{2}}(\Gamma')\ .$$

# Discrete representation



# Pressure-flux relation



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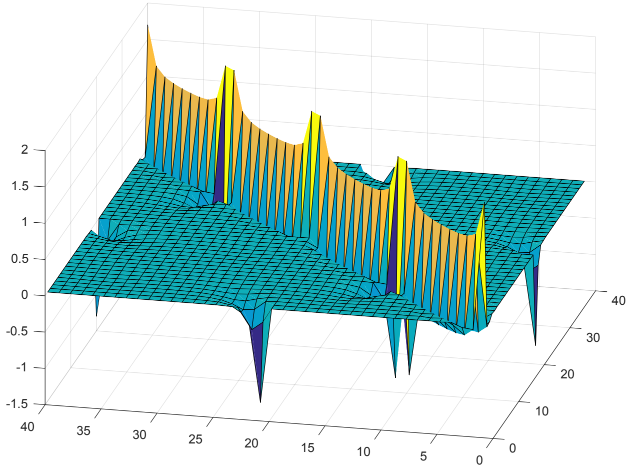

# Discrete Steklov-Poincaré matrix I

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$\Downarrow$

## Discrete Representation

$$\begin{pmatrix} \mathbb{M}_i^{(1)} & {\mathbb{E}^{d,d-1}}^T & \mathbb{N}_{i,I}^T \\ \mathbb{E}^{d,d-1} & 0 & 0 \\ \mathbb{N}_{i,I} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{u}_i \\ \mathrm{p}_i \\ \bar{\mathrm{p}} \end{pmatrix} = \begin{pmatrix} -\mathbb{N}_{i,B}^T \hat{\mathrm{p}}_i \\ \mathrm{f}_i \\ 0 \end{pmatrix}$$

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$\Downarrow$

## Discrete Steklov-Poincaré matrix

$$\mathbb{S}_i = -\mathbb{N}_{i,I}\mathbb{M}_i^{(1)^{-1}}\left(\mathbb{M}_i^{(1)} - \mathbb{E}^{d,d-1^T}\left(\mathbb{E}^{d,d-1}\mathbb{M}_i^{(1)^{-1}}\mathbb{E}^{d,d-1^T}\right)^{-1}\mathbb{E}^{d,d-1}\right)\mathbb{M}_i^{(1)^{-1}}\mathbb{N}_{i,I}^T$$

# Discrete Steklov-Poincaré matrix II

$$
\mathbb{N} =
$$

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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$$
\mathbb{E}^{d,d-1} =
$$

$$
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

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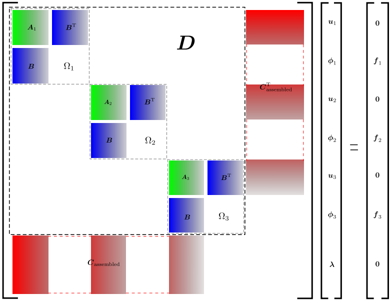
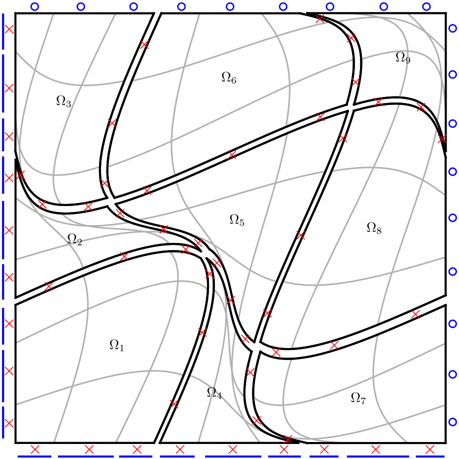
# Discrete Steklov-Poincaré matrix II

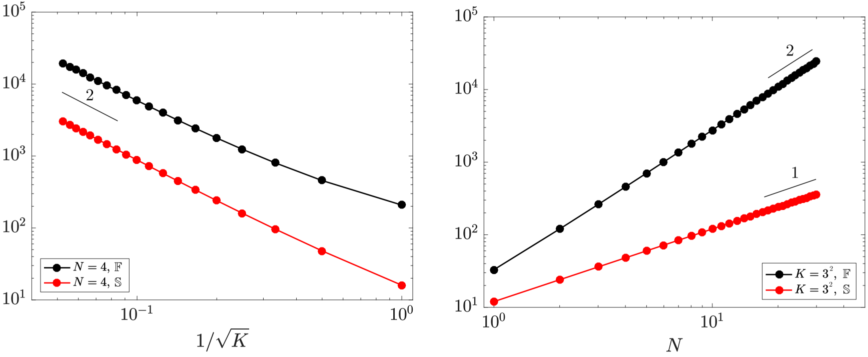
$$\begin{pmatrix} \mathbb{M}_i^{(1)} & \mathbb{E}^{d,d-1^T} & \mathbb{N}_{i,I}^T \\ \mathbb{E}^{d,d-1} & 0 & 0 \\ \mathbb{N}_{i,I} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u}_i \\ \mathsf{p}_i \\ \bar{\mathsf{p}} \end{pmatrix} = \begin{pmatrix} -\mathbb{N}_{i,B}^T \hat{\mathsf{p}}_i \\ \mathsf{f}_i \\ 0 \end{pmatrix}$$



# Numerical test



# Numerical test




Growth of the size of the full system $\mathbb{F}$ and the assembled Steklov-Poincaré matrix $\mathbb{S}$ as a function of the mesh size (left) and the polynomial degree (right).

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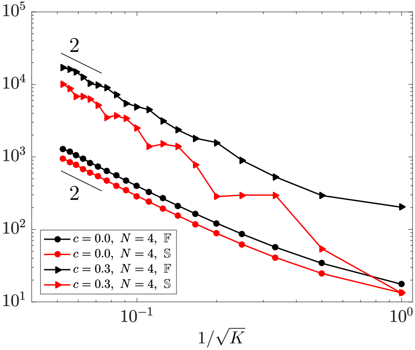
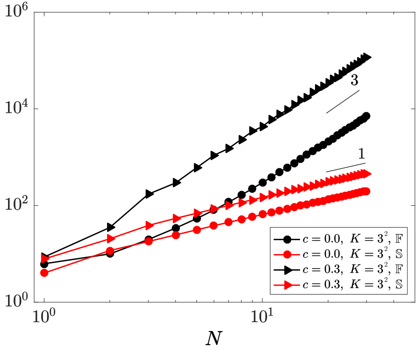
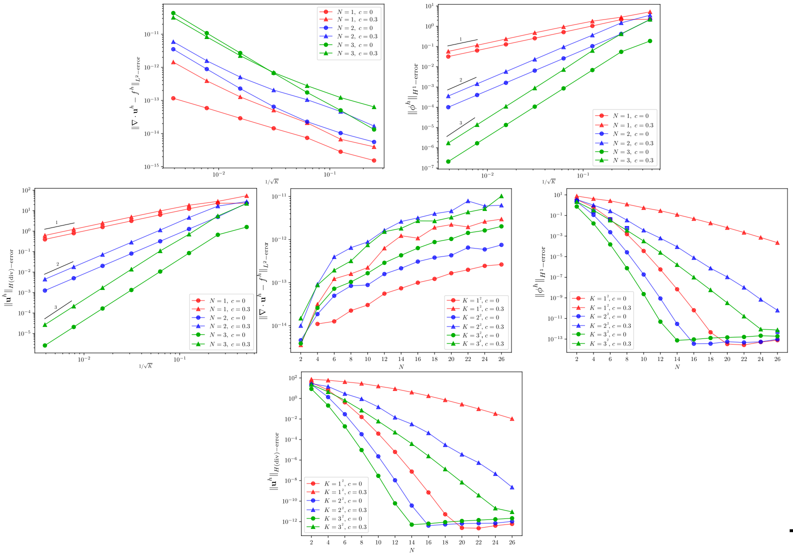
## Numerical test




Growth of the condition number of the full system $\mathbb{F}$ and the assembled Steklov-Poincaré matrix $\mathbb{S}$ as a function of the mesh size (left) and the polynomial degree (right).

# Numerical test



# Thank you

## Questions?

If you are interested, please send an email an we will send you a manuscript of this work.