

Robust and Scalable Iterative Solvers for Immersed Finite Element Methods

Frits de Prenter, Clemens Verhoosel, Harald van Brummelen,
John Evans, Joseph Benzaken & Christian Messe

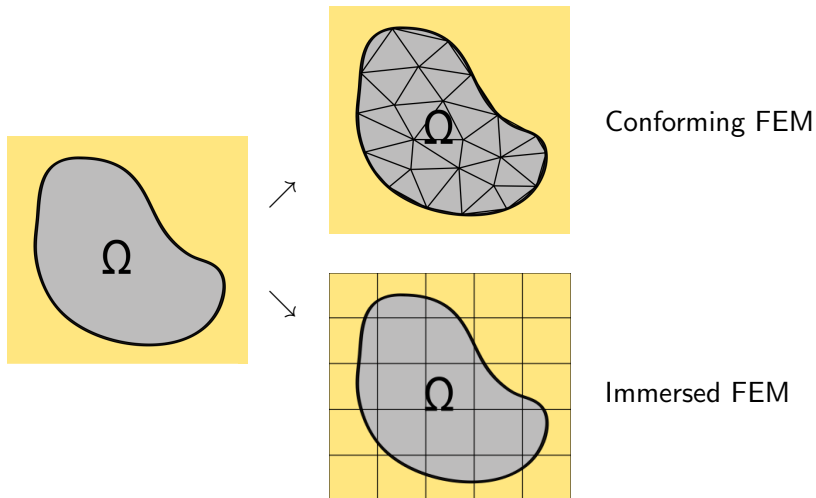


Delft, May 30th, 2018

Outline

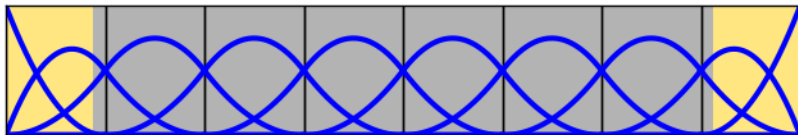
- 1 Introduction to immersed finite elements
- 2 Conditioning of immersed finite elements
- 3 Schwarz preconditioning
- 4 Implementation in multigrid cycle
- 5 Application to optimization problem
- 6 Summary and outlook

Concept of immersed methods (1): meshing

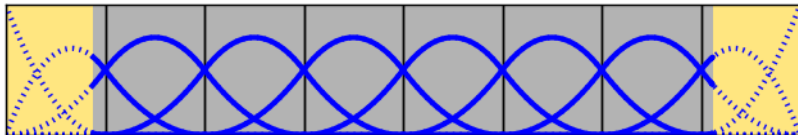


Concept of immersed methods (2): solution space

- **Step 1:** Basis functions defined on embedding domain:



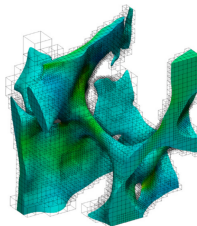
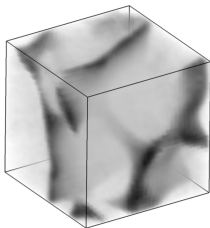
- **Step 2:** Basis functions trimmed to physical domain:



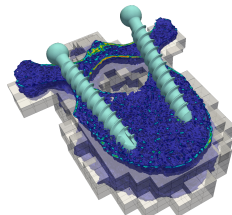
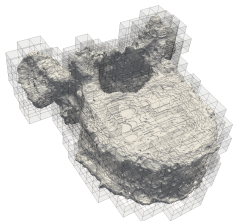
- **Step 3:** Approximation space spanned by trimmed basis functions

Application (1): elasticity on μ CT-scanned porous bone structures

- Immersogeometric analysis on trabecular bone [Verhoosel et. al. 2015]:

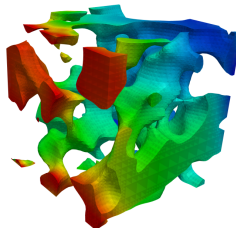
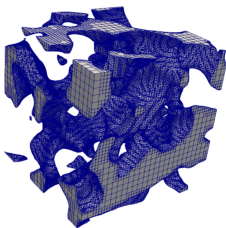


- Finite Cell Method on vertebra with implants [Elhaddad et. al. 2017]:

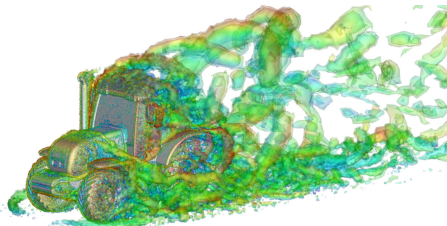
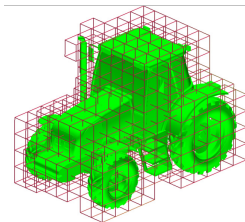


Application (2): flow on complex geometries

- Creeping flow through sinthered glass beads [*Hoang et. al. 2018*] (preprint):

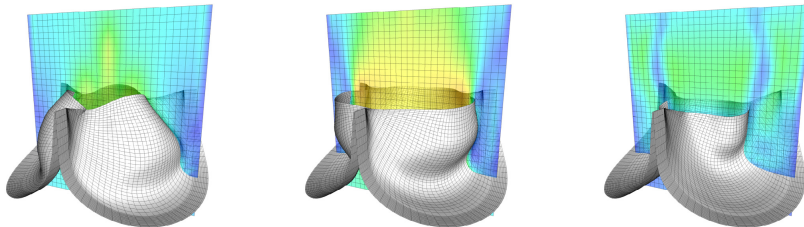


- Airflow around CAD model of a tractor [*Hsu et. al. 2016*]:

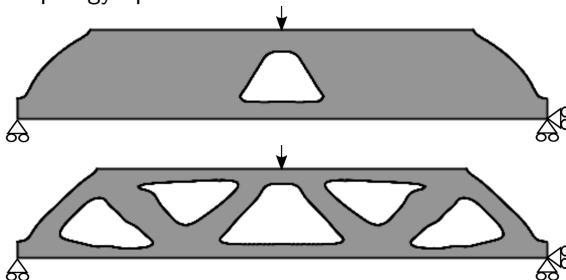


Application (3): moving domains with topology changes

- Simulation of bioprosthetic heart valves [Kamensky et. al. 2017]:

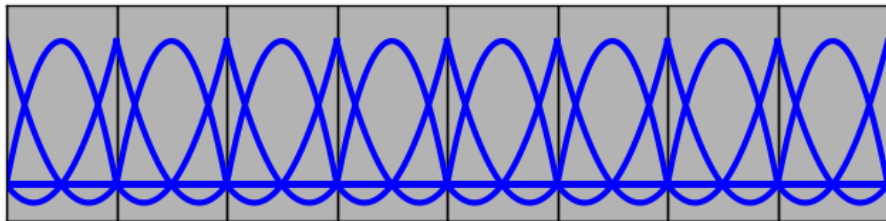


- Level set based topology optimization:



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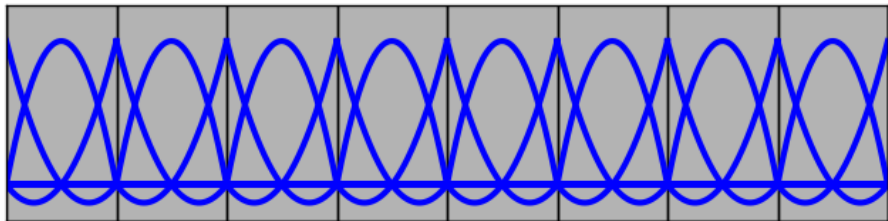
Conditioning analysis



function	v^h	$\Leftrightarrow$	$\mathbf{v}$	coefficient vector
weak form	$a(u^h, v^h) = b(v^h)$	$\Leftrightarrow$	$\mathbf{A}\mathbf{u} = \mathbf{b}$	linear system

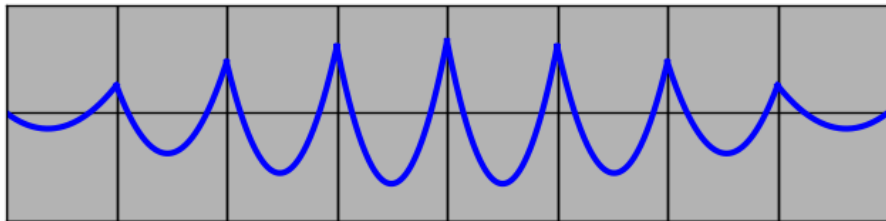
condition number: $\kappa(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$

Conditioning analysis



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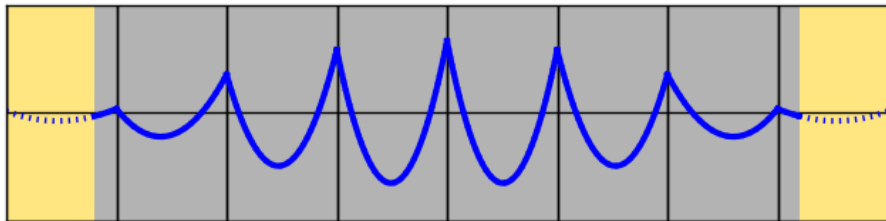
Conditioning analysis



condition number: $\kappa(\mathbf{A}) = \|\mathbf{A}\| \|\mathbf{A}^{-1}\|$

$$\|\mathbf{A}\| = \max_{\mathbf{v}} \frac{\|\mathbf{A}\mathbf{v}\|}{\|\mathbf{v}\|} = \max_{\mathbf{v}} \frac{\mathbf{v}^T \mathbf{A} \mathbf{v}}{\mathbf{v}^T \mathbf{v}} = \max_{\mathbf{v}} \frac{a(\mathbf{v}^h, \mathbf{v}^h)}{\mathbf{v}^T \mathbf{v}}$$

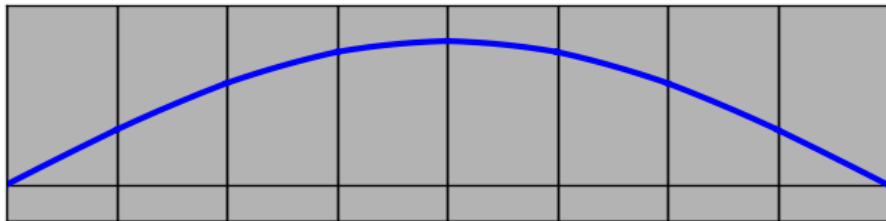
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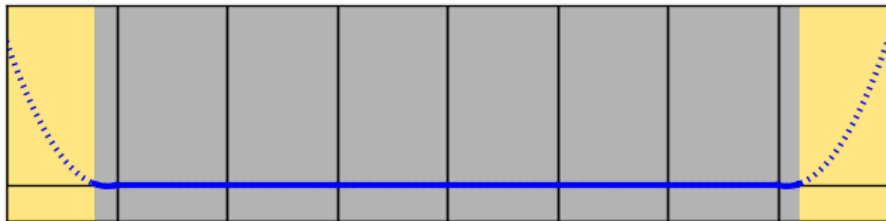
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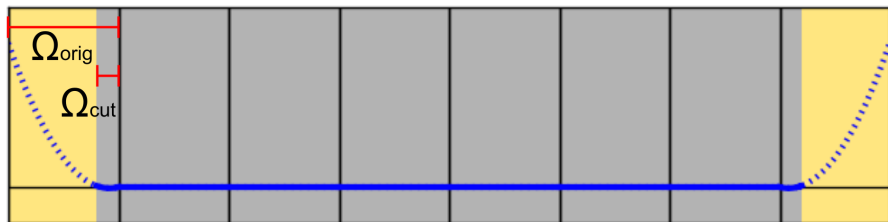
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Conditioning analysis

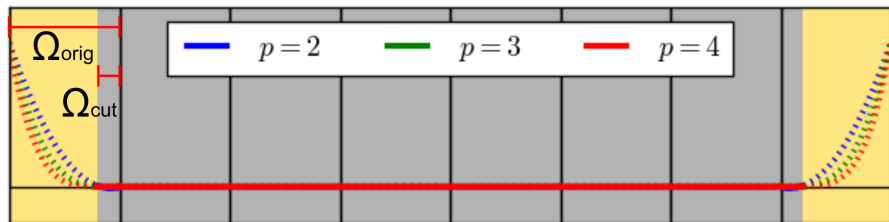


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$$\eta = \min_e \frac{|\Omega_{\text{cut}}^e|}{|\Omega_{\text{orig}}^e|} \quad \kappa \propto \eta^{-(2p+1-2/d)}$$

Conditioning analysis

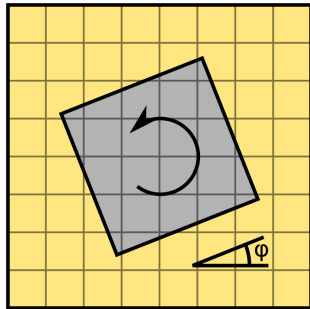
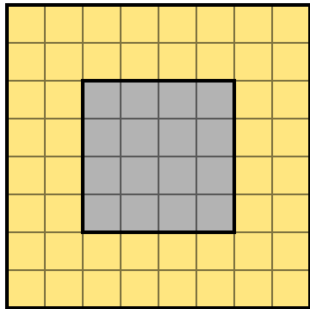


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Example (1): setup

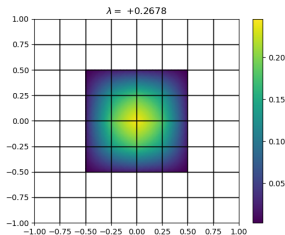
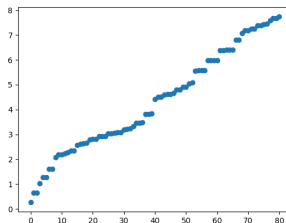


Problem setup:

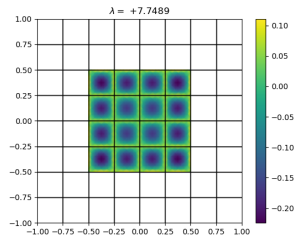
- Domain rotated over grid with $2^5 \times 2^5$ elements with a 2nd order Lagrange basis
- *Different* discretizations of the *same* problem with the *same* mesh size
- Condition number and convergence for all separate rotations

Example (2): eigenmodes

eigenvalues



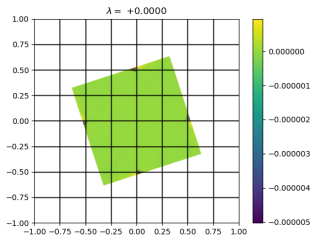
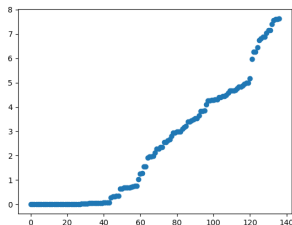
smallest eigenmode



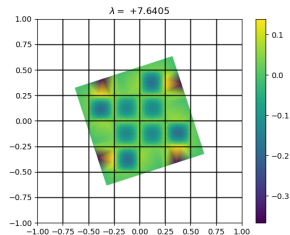
largest eigenmode

Example (2): eigenmodes

eigenvalues

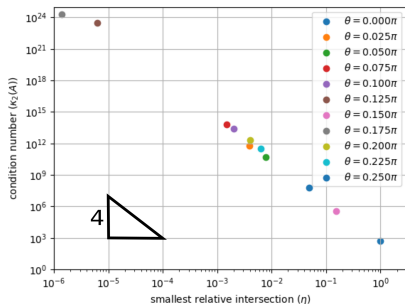


smallest eigenmode

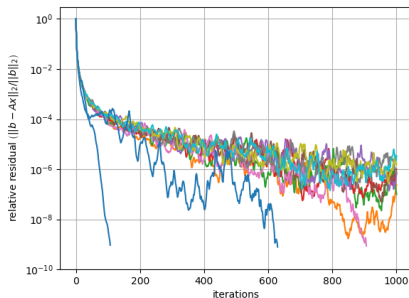


largest eigenmode

Example (3): condition numbers and convergence

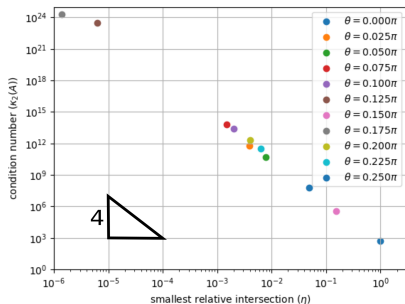


condition numbers

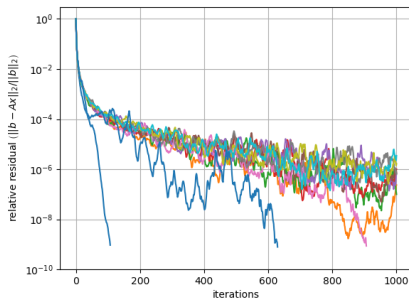


residual convergence of CG

Example (3): condition numbers and convergence



condition numbers



residual convergence of CG

Prenter, Verhoosel, Zwieteren & Brummelen: Condition number analysis and preconditioning of the finite cell method, CMAME 2017

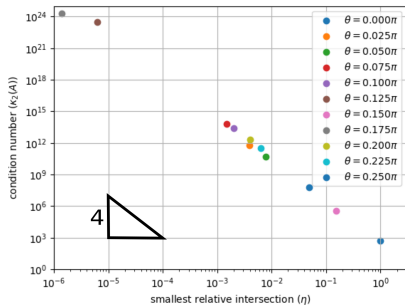
Jacobi preconditioning (1): results

unpreconditioned

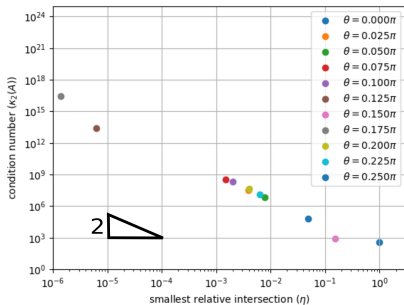
Jacobi preconditioned

Jacobi preconditioning (1): results

unpreconditioned



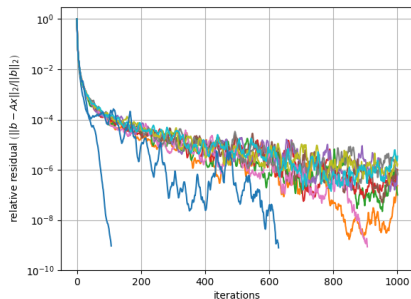
Jacobi preconditioned



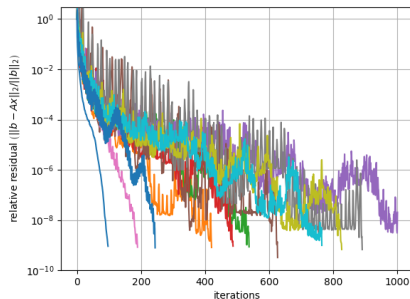
condition numbers

Jacobi preconditioning (1): results

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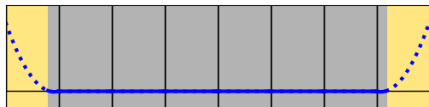
Jacobi preconditioned



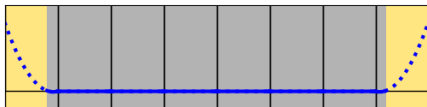
residual convergence of CG

Jacobi preconditioning (2): analysis

smallest mode
unpreconditioned

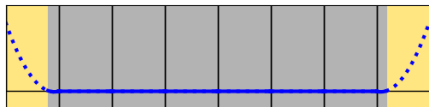


smallest mode
Jacobi preconditioned

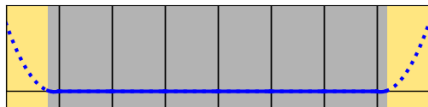


Jacobi preconditioning (2): analysis

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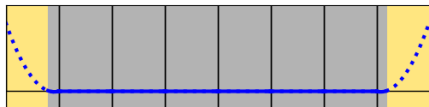
smallest mode
Jacobi preconditioned



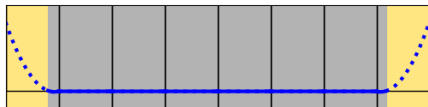
(almost) the same eigenfunction!

Jacobi preconditioning (2): analysis

smallest mode
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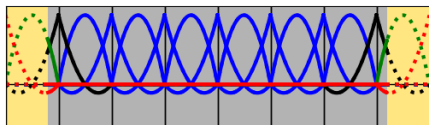


smallest mode
Jacobi preconditioned

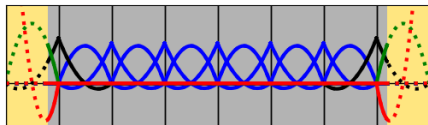


(almost) the same eigenfunction!

original basis

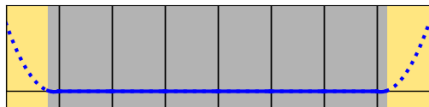


scaled basis

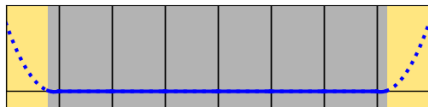


Jacobi preconditioning (2): analysis

smallest mode
unpreconditioned

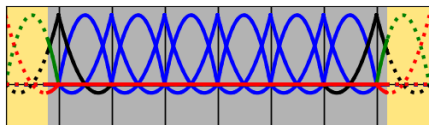


smallest mode
Jacobi preconditioned



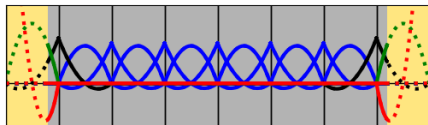
(almost) the same eigenfunction!

original basis



**basis functions small and almost
linearly dependent**

scaled basis



**basis functions scaled, but still
almost linearly dependent!**

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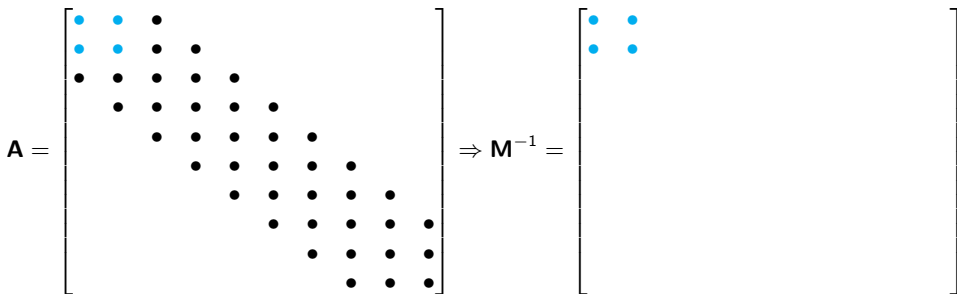
Additive-Schwarz preconditioning

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$$\mathbf{A} = \begin{bmatrix} \bullet & & & & & & & & \\ \bullet & \bullet & & & & & & & \\ \bullet & \bullet & \bullet & & & & & & \\ & \bullet & \bullet & \bullet & & & & & \\ & & \bullet & \bullet & \bullet & & & & \\ & & & \bullet & \bullet & \bullet & & & \\ & & & & \bullet & \bullet & \bullet & & \\ & & & & & \bullet & \bullet & \bullet & \\ & & & & & & \bullet & \bullet & \bullet \\ & & & & & & & \bullet & \bullet & \bullet \end{bmatrix} \Rightarrow \mathbf{M}^{-1} = \begin{bmatrix} & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{bmatrix}$$

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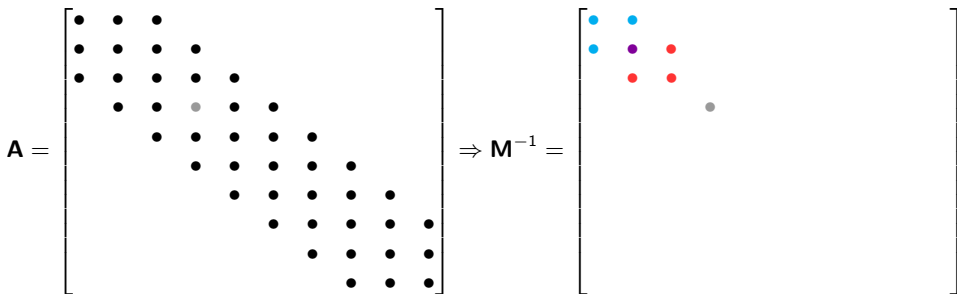
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The diagram illustrates the Additive-Schwarz preconditioning process. On the left, matrix $\mathbf{A}$ is shown as a sparse matrix with a block-diagonal structure. The non-zero entries are represented by black dots, with some entries in the upper-left block highlighted in red. An arrow points to the right, where matrix $\mathbf{M}^{-1}$ is shown. Matrix $\mathbf{M}^{-1}$ is a preconditioner matrix, and its non-zero entries are highlighted in different colors: cyan, purple, and red, indicating the contribution of different subproblems.

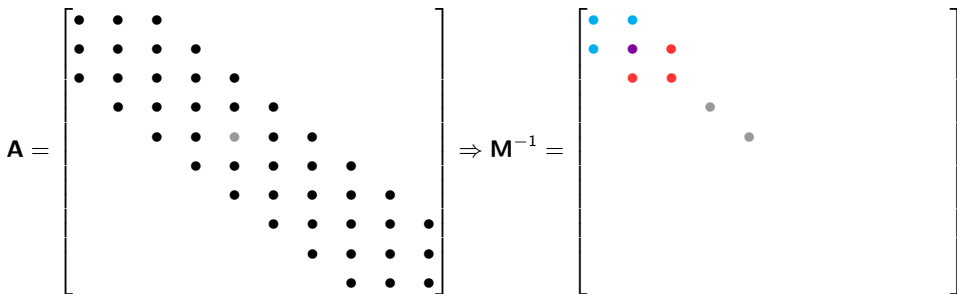
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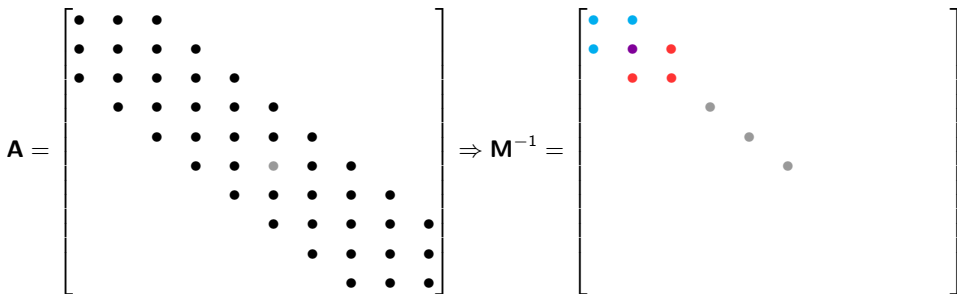
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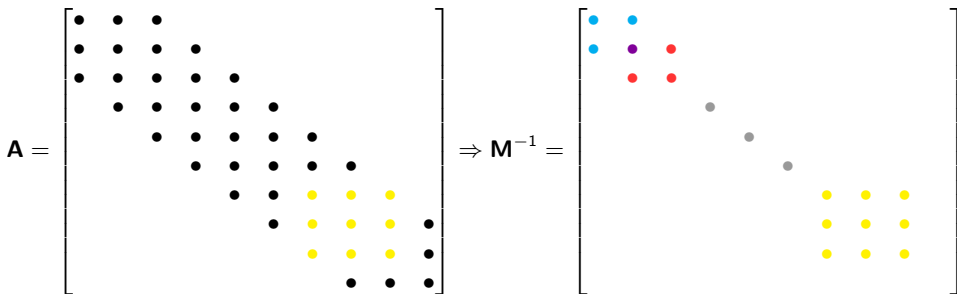
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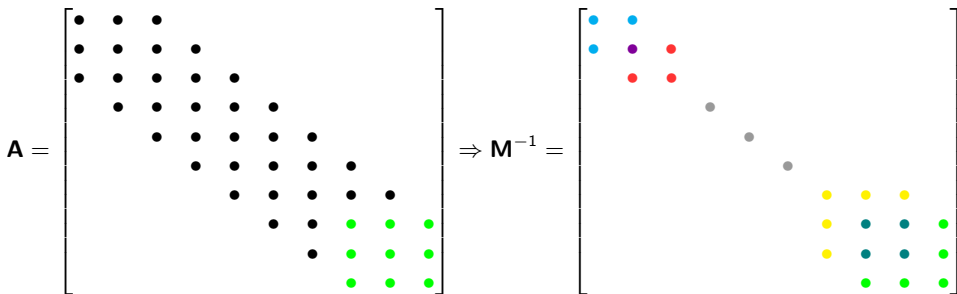
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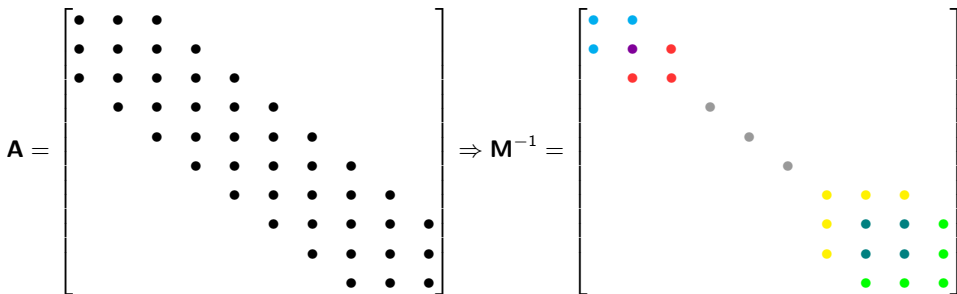
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$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)}_{\mathbf{A}_i}^{-1} \mathbf{P}_i$$



Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$



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$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)}_{\mathbf{A}_i}^{-1} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

$$\mathbf{v}_1 = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \Rightarrow \mathbf{P}_1^T \mathbf{v}_1 = \begin{bmatrix} \bullet \\ \bullet \\ \\ \\ \\ \end{bmatrix} \Rightarrow \sum_{i=1}^{i=0} \mathbf{P}_i^T \mathbf{v}_i = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

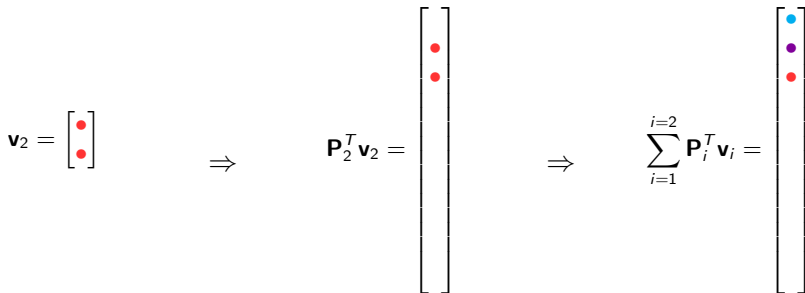
$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

$$\mathbf{v}_1 = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \Rightarrow \mathbf{P}_1^T \mathbf{v}_1 = \begin{bmatrix} \bullet \\ \bullet \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} \Rightarrow \sum_{i=1} \mathbf{P}_i^T \mathbf{v}_i = \begin{bmatrix} \bullet \\ \bullet \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

$$\mathbf{v}_2 = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \Rightarrow \mathbf{P}_2^T \mathbf{v}_2 = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix} \Rightarrow \sum_{i=1}^{i=2} \mathbf{P}_i^T \mathbf{v}_i = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix}$$


Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

$$\mathbf{v}_3 = [\bullet]$$

 $\Rightarrow$

$$\mathbf{P}_3^T \mathbf{v}_3 = \begin{bmatrix} \bullet \\ \vdots \\ \bullet \\ \vdots \end{bmatrix}$$

 $\Rightarrow$

$$\sum_{i=1}^{i=3} \mathbf{P}_i^T \mathbf{v}_i = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

$$\mathbf{v}_4 = [\bullet]$$

 $\Rightarrow$

$$\mathbf{P}_4^T \mathbf{v}_4 = \begin{bmatrix} \bullet \\ \\ \\ \\ \end{bmatrix}$$

 $\Rightarrow$

$$\sum_{i=1}^{i=4} \mathbf{P}_i^T \mathbf{v}_i = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

$$\mathbf{v}_5 = [\bullet]$$

 $\Rightarrow$

$$\mathbf{P}_5^T \mathbf{v}_5 = \begin{bmatrix} \bullet \\ \vdots \\ \bullet \\ \vdots \end{bmatrix}$$

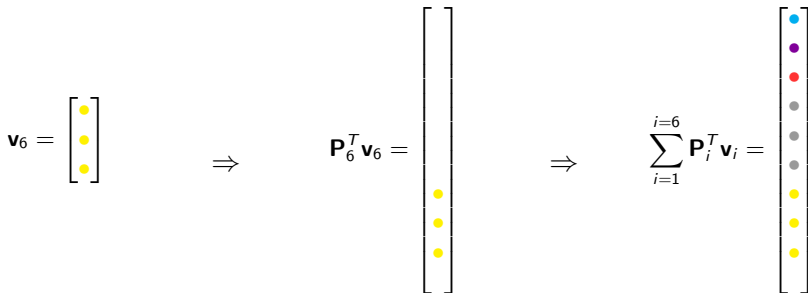
 $\Rightarrow$

$$\sum_{i=1}^5 \mathbf{P}_i^T \mathbf{v}_i = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \vdots \end{bmatrix}$$

Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

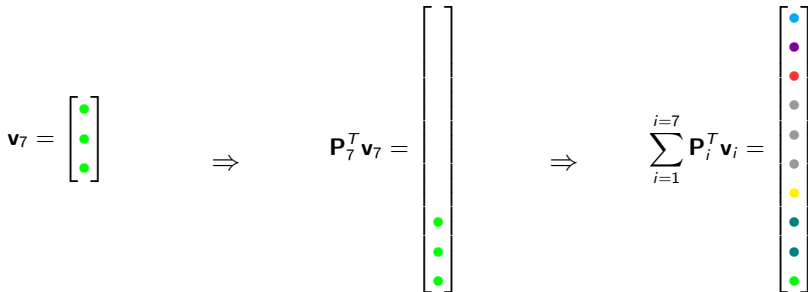
$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$



Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$



Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

Additive-Schwarz lemma:

$$\mathbf{v}^T \mathbf{M} \mathbf{v} = \min_{\sum_i \mathbf{P}_i^T \mathbf{v}_i = \mathbf{v}} \sum_i \mathbf{v}_i^T \mathbf{A}_i \mathbf{v}_i$$

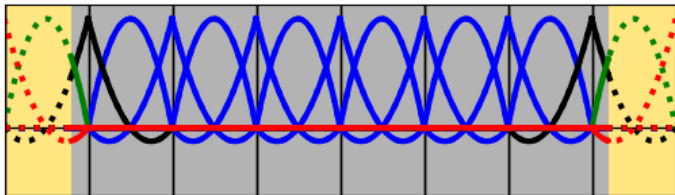
Additive-Schwarz preconditioning

$$\mathbf{M}^{-1} = \sum_i \mathbf{P}_i^T \underbrace{(\mathbf{P}_i \mathbf{A} \mathbf{P}_i^T)^{-1}}_{\mathbf{A}_i} \mathbf{P}_i$$

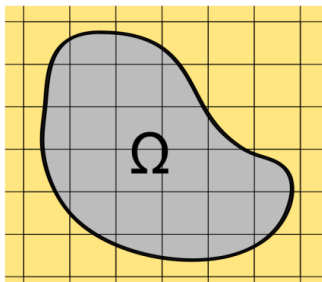
$$\mathbf{v} = \sum_i \mathbf{P}_i^T \mathbf{v}_i$$

Additive-Schwarz lemma:

$$\mathbf{v}^T \mathbf{M} \mathbf{v} = \min_{\sum_i \mathbf{P}_i^T \mathbf{v}_i = \mathbf{v}} \sum_i \mathbf{v}_i^T \mathbf{A}_i \mathbf{v}_i$$

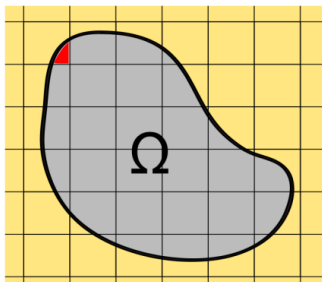


Setting blocks



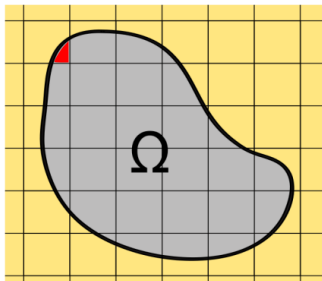
$$\mathbf{A} = \begin{bmatrix} \text{block} & & \\ & \text{block} & \\ & & \text{block} \end{bmatrix}$$
$$\mathbf{M}^{-1} = \begin{bmatrix} \text{block} & & \\ & \text{block} & \\ & & \text{block} \end{bmatrix}$$

Setting blocks



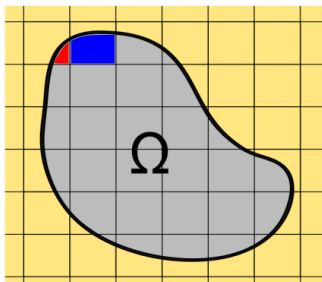
$$\mathbf{A} = \begin{bmatrix} \text{black diagonal band} \end{bmatrix}$$
$$\mathbf{M}^{-1} = \begin{bmatrix} \text{white} \end{bmatrix}$$

Setting blocks



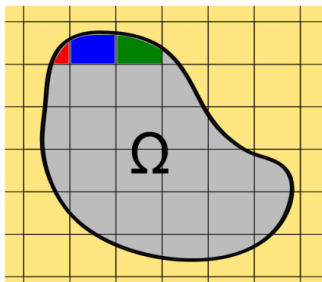
The diagram illustrates the structure of two matrices, A and M^{-1} , used in the context of the conjugate gradient method. Matrix A is shown as a banded matrix with a red square on the diagonal, indicating it is symmetric and positive definite. Matrix M^{-1} is shown as a sparse matrix with a red square on the diagonal, indicating it is symmetric and positive definite.

Setting blocks



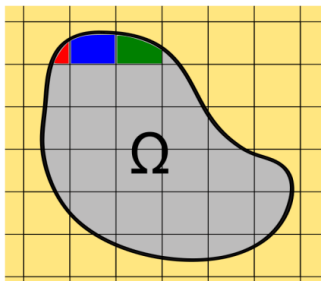
$$\mathbf{A} = \begin{bmatrix} \text{black diagonal band} \\ \text{red square} \\ \text{blue square} \end{bmatrix}$$
$$\mathbf{M}^{-1} = \begin{bmatrix} \text{red square} \\ \text{blue square} \end{bmatrix}$$

Setting blocks



$$\mathbf{A} = \begin{bmatrix} \text{black} & & & & \\ & \text{red} & & & \\ & & \text{blue} & & \\ & & & \text{green} & \\ & & & & \text{black} & & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ & & & & & & & \ddots & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \text{black} \end{bmatrix}$$
$$\mathbf{M}^{-1} = \begin{bmatrix} & & & & \\ & \text{red} & & & \\ & & \text{blue} & & \\ & & & \text{green} & \\ & & & & \end{bmatrix}$$

Setting blocks



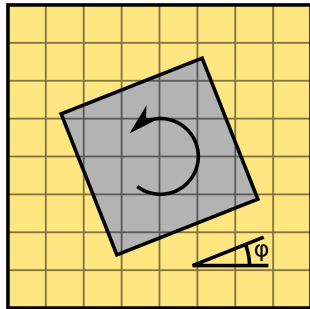
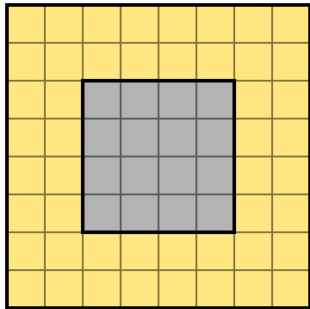
$$\mathbf{A} = \begin{bmatrix} \text{diagonal blocks} \\ \text{off-diagonal blocks} \end{bmatrix}$$

The matrix $\mathbf{A}$ is represented as a block matrix. The diagonal blocks are colored red, blue, and green, corresponding to the blocks in the region Ω . The off-diagonal blocks are black, representing the coupling between the blocks.

$$\mathbf{M}^{-1} = \begin{bmatrix} \text{diagonal blocks} \\ \text{off-diagonal blocks} \end{bmatrix}$$

The matrix $\mathbf{M}^{-1}$ is represented as a block matrix. The diagonal blocks are colored red, blue, and green, corresponding to the blocks in the region Ω . The off-diagonal blocks are black, representing the coupling between the blocks.

Example revisited (1): setup

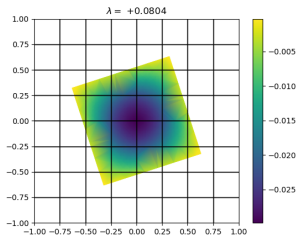
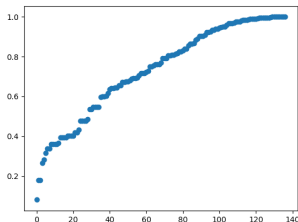


Problem setup:

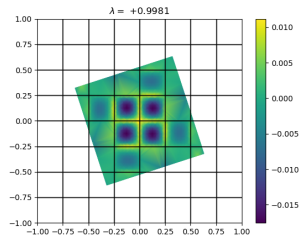
- Domain rotated over grid with $2^n \times 2^n$ elements with a 2^{nd} order Lagrange basis
- *Different* discretizations of the *same* problem with the *same* mesh size
- Condition number and convergence for **the preconditioned system**

Example revisited (2): eigenmodes

eigenvalues



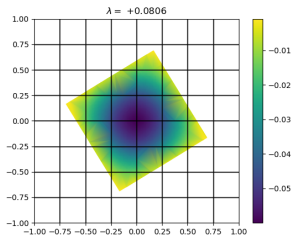
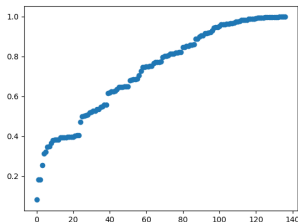
smallest eigenmode



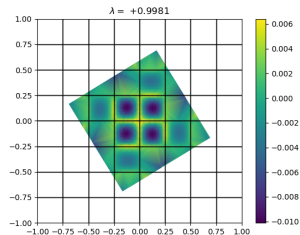
largest eigenmode

Example revisited (2): eigenmodes

eigenvalues



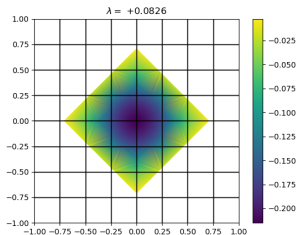
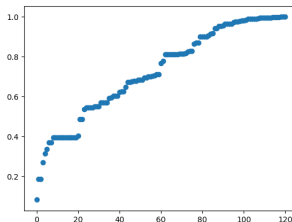
smallest eigenmode



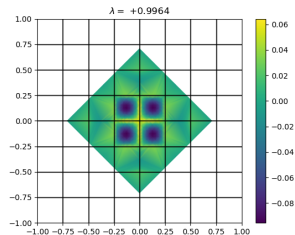
largest eigenmode

Example revisited (2): eigenmodes

eigenvalues

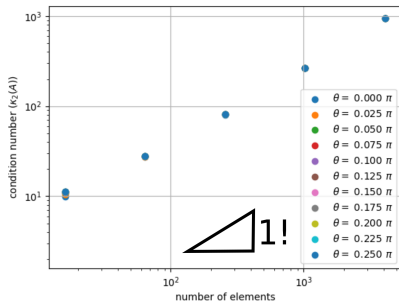


smallest eigenmode



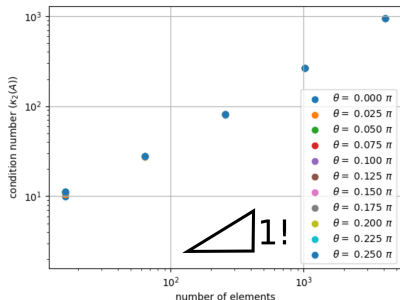
largest eigenmode

Example revisited (3): condition numbers and convergence

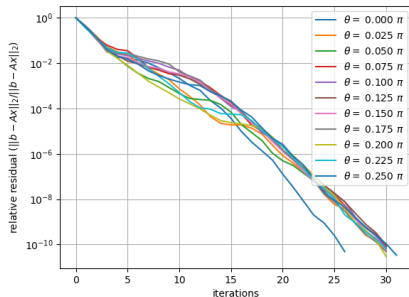


condition numbers

Example revisited (3): condition numbers and convergence

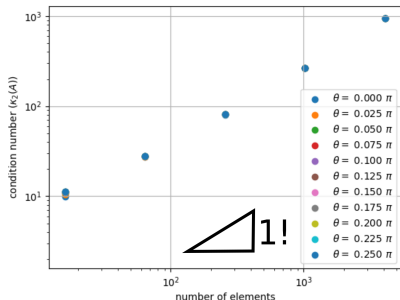


condition numbers

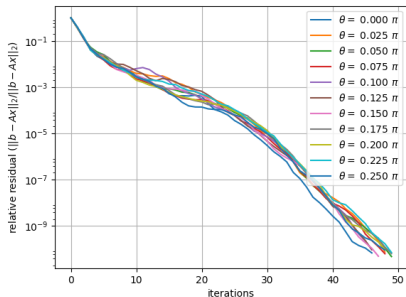


residual convergence of CG
 $2^3 \times 2^3$ elements

Example revisited (3): condition numbers and convergence

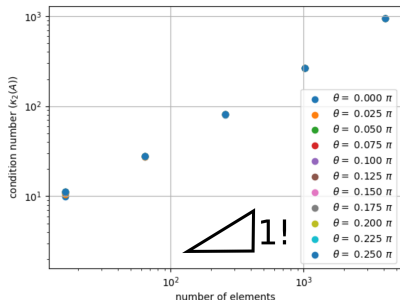


condition numbers

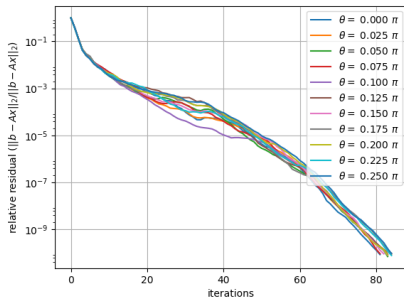


residual convergence of CG
 $2^4 \times 2^4$ elements

Example revisited (3): condition numbers and convergence

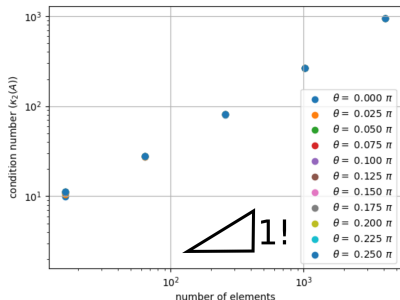


condition numbers

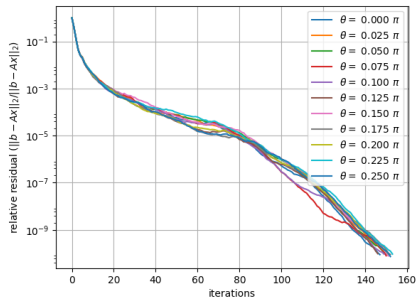


residual convergence of CG
 $2^5 \times 2^5$ elements

Example revisited (3): condition numbers and convergence



condition numbers



residual convergence of CG
 $2^6 \times 2^6$ elements

Outline

- 1 Introduction to immersed finite elements
- 2 Conditioning of immersed finite elements
- 3 Schwarz preconditioning
- 4 Implementation in multigrid cycle**
- 5 Application to optimization problem
- 6 Summary and outlook



Thank you Prof. Oosterlee!



Thank you Prof. Oosterlee!



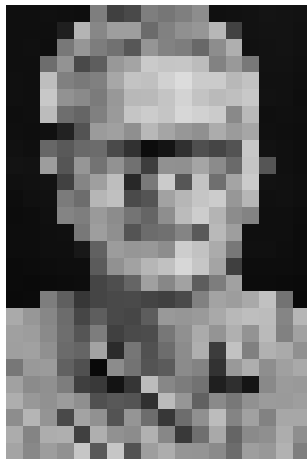
Thank you Prof. Oosterlee!



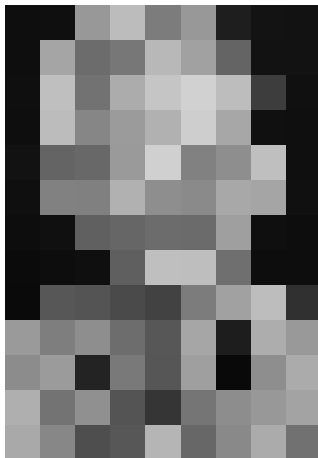
Thank you Prof. Oosterlee!



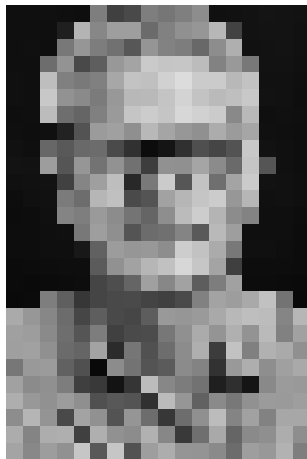
Thank you Prof. Oosterlee!



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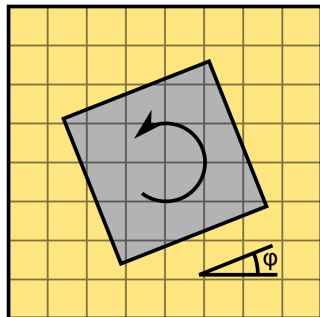


Thank you Prof. Oosterlee!

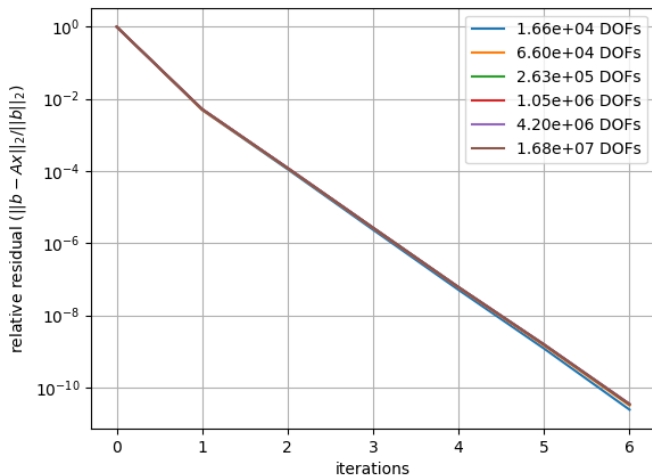
Example (1): problem setup

Problem setup:

- Domain at $\phi = 22.5^\circ$ with background grid
- $2^n \times 2^n$ elements for $n \in \{6, 7, 8, 9, 10, 11\}$
- $\{3, 4, 5, 6, 7, 8\}$ levels
- 2nd order Lagrange basis
- $\approx \frac{1}{4}$ of basis functions active
- Preconditioned CG solver
- V-cycle smoothed with Multiplicative-Schwarz/
Gauss-Seidel scheme

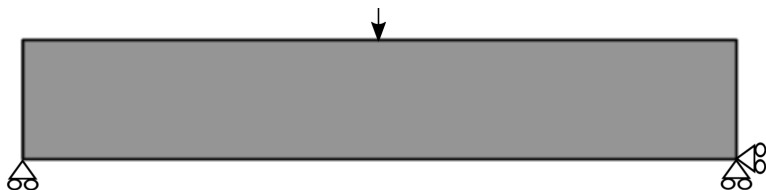


Example (2): convergence results



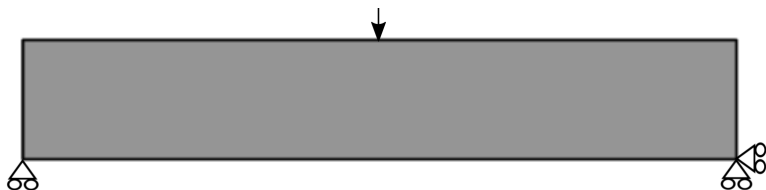
- 1 Introduction to immersed finite elements
- 2 Conditioning of immersed finite elements
- 3 Schwarz preconditioning
- 4 Implementation in multigrid cycle
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Level set based optimization of beam

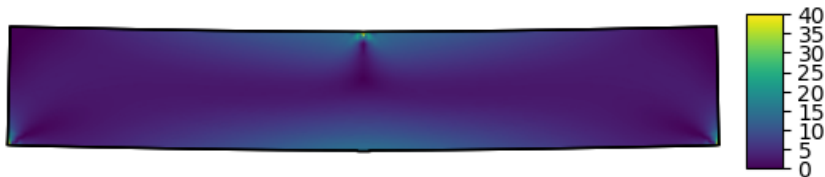


initial geometry

Level set based optimization of beam

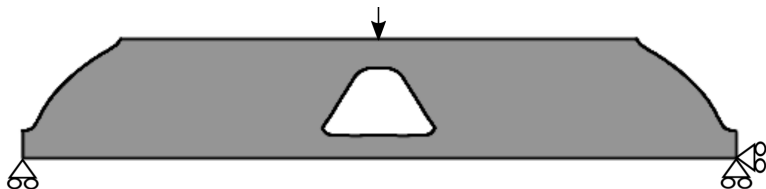


initial geometry

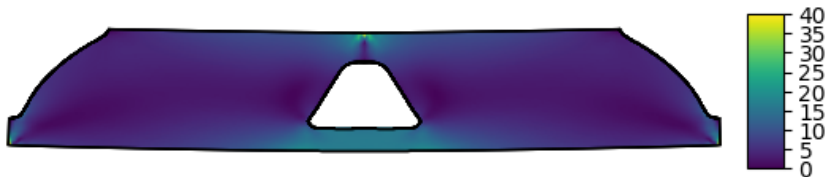


displacement and stress

Level set based optimization of beam

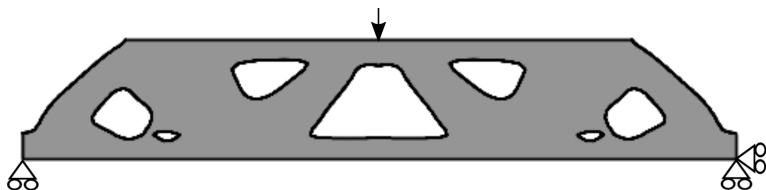


geometry after 30 iterations

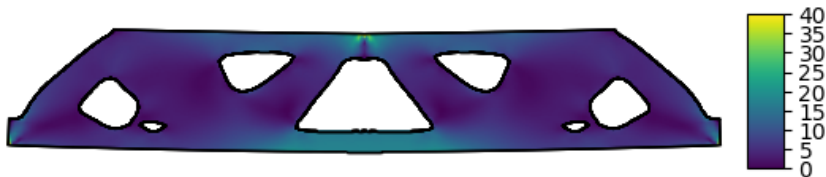


displacement and stress

Level set based optimization of beam

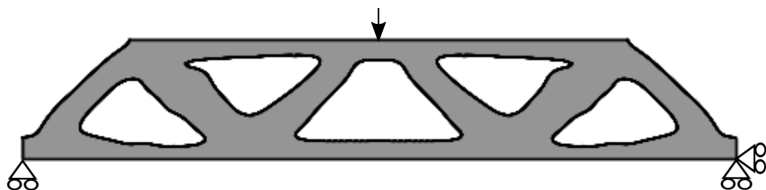


geometry after 40 iterations

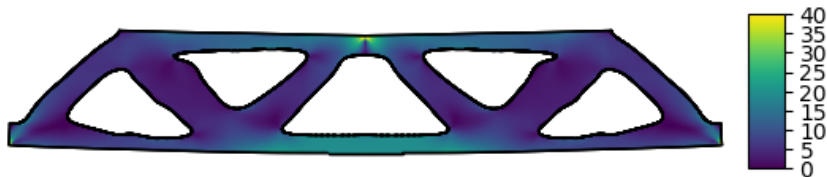


displacement and stress

Level set based optimization of beam

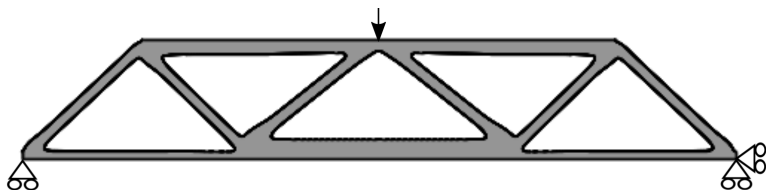


geometry after 50 iterations

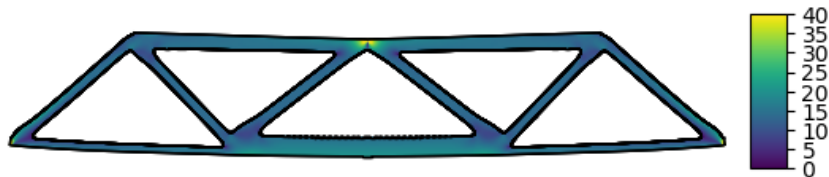


displacement and stress

Level set based optimization of beam

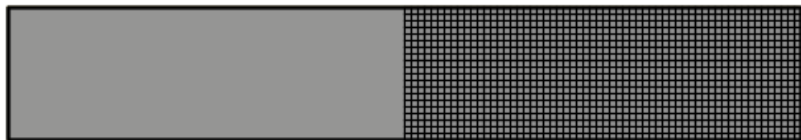


geometry after 150 iterations



displacement and stress

Immersed discretization

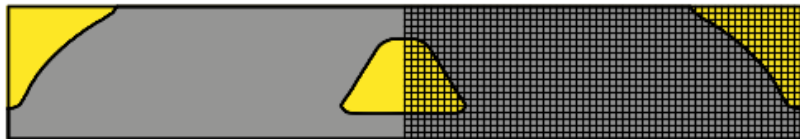


initial geometry and mesh

Mesh details:

- Half of the problem solved using symmetry in the vertical plane
- $\{60 \times 20, 120 \times 40, 240 \times 80\}$ elements at coarsest levels
- $\{2, 3, 4\}$ levels of coarsening
- 1 level of refinement after 50 iterations
- 2 levels of refinement after 125 iterations
- Truncated hierarchical quadratic B-splines

Immersed discretization

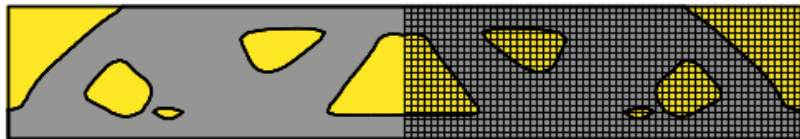


geometry and mesh after 30 iterations

Mesh details:

- Half of the problem solved using symmetry in the vertical plane
- $\{60 \times 20, 120 \times 40, 240 \times 80\}$ elements at coarsest levels
- $\{2, 3, 4\}$ levels of coarsening
- 1 level of refinement after 50 iterations
- 2 levels of refinement after 125 iterations
- Truncated hierarchical quadratic B-splines

Immersed discretization

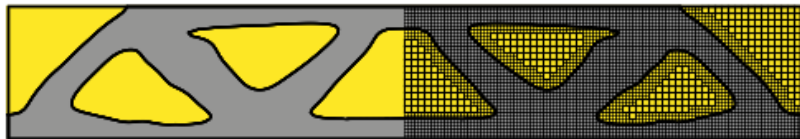


geometry and mesh after 40 iterations

Mesh details:

- Half of the problem solved using symmetry in the vertical plane
- $\{60 \times 20, 120 \times 40, 240 \times 80\}$ elements at coarsest levels
- $\{2, 3, 4\}$ levels of coarsening
- 1 level of refinement after 50 iterations
- 2 levels of refinement after 125 iterations
- Truncated hierarchical quadratic B-splines

Immersed discretization

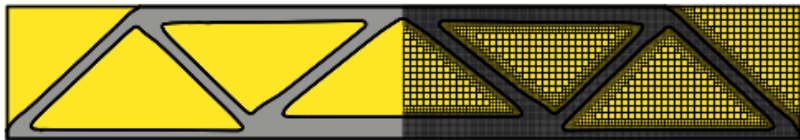


geometry and mesh after 50 iterations

Mesh details:

- Half of the problem solved using symmetry in the vertical plane
- $\{60 \times 20, 120 \times 40, 240 \times 80\}$ elements at coarsest levels
- $\{2, 3, 4\}$ levels of coarsening
- 1 level of refinement after 50 iterations
- 2 levels of refinement after 125 iterations
- Truncated hierarchical quadratic B-splines

Immersed discretization

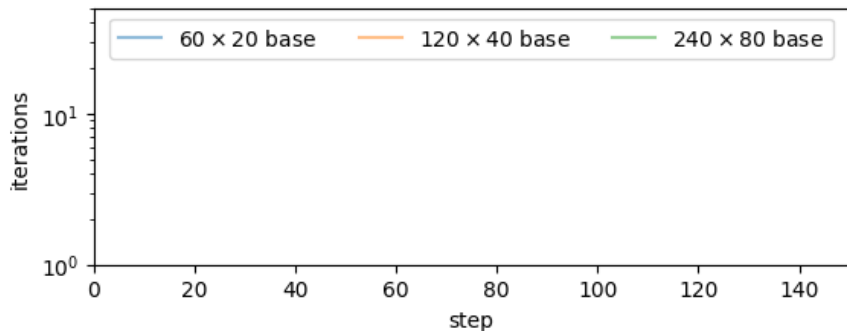
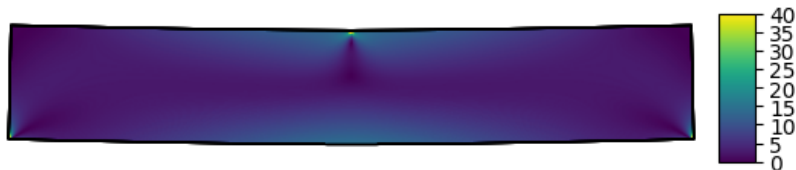


geometry and mesh after 150 iterations

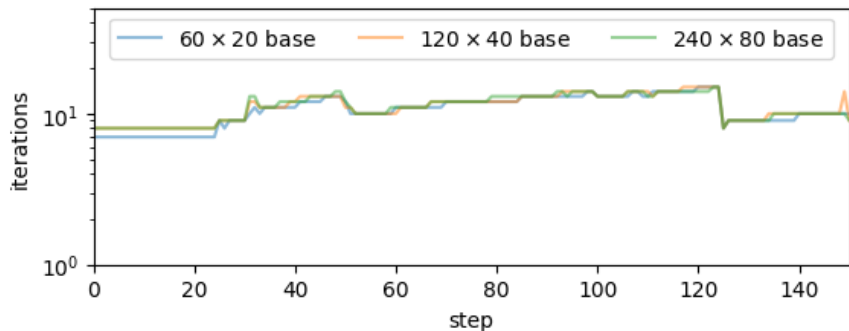
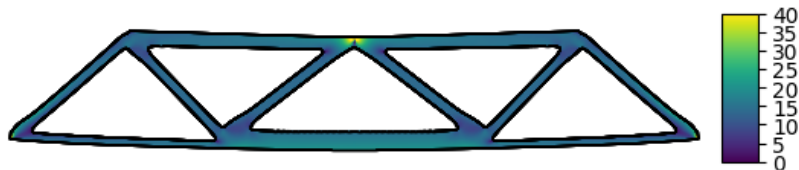
Mesh details:

- Half of the problem solved using symmetry in the vertical plane
- $\{60 \times 20, 120 \times 40, 240 \times 80\}$ elements at coarsest levels
- $\{2, 3, 4\}$ levels of coarsening
- 1 level of refinement after 50 iterations
- 2 levels of refinement after 125 iterations
- Truncated hierarchical quadratic B-splines

Iteratively solving the linear systems



Iteratively solving the linear systems



Outline

- 1 Introduction to immersed finite elements
- 2 Conditioning of immersed finite elements
- 3 Schwarz preconditioning
- 4 Implementation in multigrid cycle
- 5 Application to optimization problem
- 6 Summary and outlook**

Summary

- Preconditioner for immersed finite elements that is robust to:
 - how elements are cut
 - the mesh size

Outlook

- Testing on three-dimensional testcases and scanned data
- Extending the procedure for fluid problems

Robust and Scalable Iterative Solvers for Immersed Finite Element Methods

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John Evans, Joseph Benzaken & Christian Messe



Delft, May 30th, 2018