Market risk measures with stochastic liquidity horizon by Shannon wavelet expansions

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Outline

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Motivation

Risk measures

Definition

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a time horizon Δt.
Denote by $\mathcal{L}$ the set of all random variables on $(\Omega, \mathcal{F})$ (representing the portfolio returns/loses over a time horizon Δt).
Then, **risk measures** are real-valued maps $\rho : \mathcal{L} \to \mathbb{R}$.

A risk measure is **coherent** if it satisfies: normality, monotonicity, sub-additivity, positive homogeneity and translation invariance.

- **Use**: determine the amount of currency to keep in reserve.
- **Purpose**: make the risks taken by financial institutions $\left\{ \begin{array}{l} \text{banks} \\ \text{insurance companies} \end{array} \right.$ acceptable to the regulator.
- The most famous: **VaR** and **ES**.

Value at Risk (VaR) and Expected Shortfall (ES)

Definition

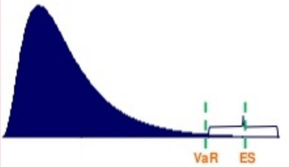
Given a confidence level $\alpha \in (0, 1)$. Being L a loss.

The **VaR**$_\alpha$ is the smallest number l such that the probability that the loss L exceeds l is at most $(1 - \alpha)$. I.e.

$$\mathrm{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1-\alpha\}.$$

The **ES**$_\alpha$ is defined by

$$\mathrm{ES}_\alpha(L) := \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(L)\, du.$$



- VaR is a quantile of the loss distribution.
- VaR is not a coherent risk measure, not satisfies sub-additivity.
- ES is more sensitive to the shape of the loss distribution in the tail of the distribution. ES is a coherent risk measure.

The financial crisis, review of BCBS

Basel Committee of Banking Supervision (BCBS) stated:

- The crisis exposed:
  - Weaknesses in the framework design for capitalizing trading activities.
  - Insufficient capital level required against trading book exposures to absorb losses.

- Review/assessment:
  - From VaR to ES, due to the inability to capture the risk in the tail.
  - Incorporate market liquidity risk. The time it takes to liquidate a risk position varies; thus, the horizon should be extended.

Our purpose

- Produce a **set of numerical techniques** to address the challenge of the VaR and ES computation under a stochastic liquidity horizon framework (idea from Brigo and Nordio, 2015).

- To do so, we use **SWIFT**. Because:
  - In the scenarios we work, the **characteristic function** of the density is know. Thus, makes sense a Fourier inversion method.
  - Densities with stochastic holding period have fat tails, so we do no need to relay on a **truncation range**.
  - Make use of **wavelets** properties to get the risk measures values.
  - Analysis of the **error** is available.
  - There are rules on how to select the **parameters**.

SWIFT
Shannon Wavelet Inverse Fourier Technique

Multiresolution analysis (1)

Definition

Let V_j, $j = \cdots, -2, -1, 0, 1, 2, \cdots$ be a sequence of subspaces of functions in $L^2(\mathbb{R})$. The collection of spaces $(V_j)_{j\in\mathbb{Z}}$ is called a **multiresolution analysis (MRA)** of $L^2(\mathbb{R})$ with scaling function $\phi \in V_0$, if the following conditions hold

1. *(nested)* $V_j \subset V_{j+1}$,
2. *(dense)* $\overline{\cup V_j} = L^2(\mathbb{R})$,
3. *(separation)* $\cap V_j = \{0\}$,
4. *(scaling)* The function $f(x)$ belongs to V_j if and only if the function $f(2x)$ belongs to V_{j+1},
5. *(orthonormal basis)* The function ϕ belongs to V_0 and the set $\{\phi(x-k), k \in \mathbb{Z}\}$ is an orthonormal basis (using the L^2 inner product) for V_0.

MRA defines general wavelet structures in $L^2(\mathbb{R})$.

Multiresolution analysis (2)

- The set of functions

$$\{\phi_{m,k}(x) = 2^{m/2}\phi(2^m x - k); k \in \mathbb{Z}\}$$

is an orthonormal basis for V_m.

Lemma

Let us define $\mathcal{P}_m f$ as the ***orthogonal projection*** *of a function $f \in L^2(\mathbb{R})$ on the space V_m, constructed by*

$$\mathcal{P}_m f(x) = \sum_{k\in\mathbb{Z}} c_{m,k}\phi_{m,k}(x),$$

where $c_{m,k} = \int_{\mathbb{R}} f(x)\bar{\phi}(x)dx$. Then, the ***convergence*** *of the projection $\mathcal{P}_m f(x)$ holds in the $L^2(\mathbb{R}) -$ norm.*

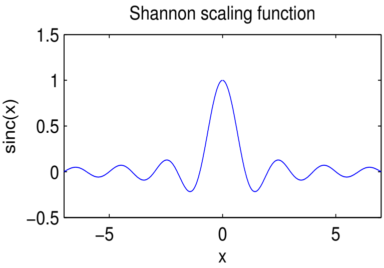
Shannon wavelets

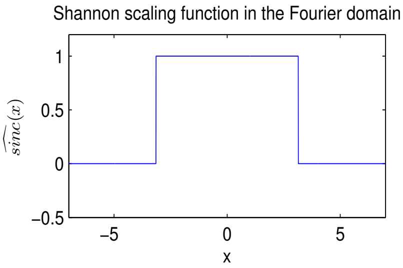
- Cardinal sine function (sinc):

$$\phi(x) = \text{sinc}(x) := \frac{\sin(\pi x)}{\pi x} \quad \text{(Shannon scaling function).}$$

- Simplicity in the Fourier domain:

$$\hat{\phi}(\omega) := \int_{\mathbb{R}} \phi(x) \mathrm{e}^{-i\omega x} dx = \text{rect}\left(\frac{\omega}{2\pi}\right).$$





SWIFT (Shannon Wavelet Inverse Fourier Technique)

SWIFT (From: Ortiz-Gracia and Oosterlee, 2015)

Let us consider a density function $f \in L^2(\mathbb{R})$. Assuming $\hat{f}$ to be known. Following MRA we approximate f by f_m:

$$f(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c^*_{m,k}\phi_{m,k}(x),$$

where $c^*_{m,k} \approx c_{m,k} = \langle f, \phi_{m,k} \rangle$ (scaling coefficients).

Approximation technique:

- Step 1: Projection on the space V_m (seen).
- Step 2: Truncation of the infinite sum.
- Step 3: Approximation of the scaling coefficients by assuming known the characteristic function of f.

SWIFT (Sum truncation (1))

Lemma

The scaling coefficients $c_{m,k}$ satisfy,

$$\lim_{k \to \pm\infty} c_{m,k} = 0.$$

Proof.

The set of Shannon scaling functions in V_m is defined as

$$\phi_{m,k}(x) = 2^{m/2}\frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)}, \quad k \in \mathbb{Z}.$$

Thus, for $h \in \mathbb{Z}$,

$$\phi_{m,k}\left(\frac{h}{2^m}\right) = 2^{m/2}\delta_{k,h},$$

being $\delta_{k,h}$ the Kronecker delta.
It gives us that

$$\mathcal{P}_m f\left(\frac{h}{2^m}\right) = 2^{m/2}\sum_{k\in\mathbb{Z}} c_{m,k}\delta_{k,h} = 2^{m/2}c_{m,h}.$$

Since f is a density function, we assume it to tend to zero at plus and minus infinity. □

SWIFT (Coefficients approximation)

To do so, we make use of Vieta's formula.

Vieta's formula

$$\operatorname{sinc}(t) := \frac{\sin(\pi t)}{(\pi t)} \approx \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos\left(\frac{2j-1}{2^J} \pi t\right).$$

Using Vieta's formula and some algebraic manipulation, one arrives to the coefficients expression.

Coefficients approximation

$$c_{m,k} \approx c^*_{m,k} := \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \operatorname{Re}\left[\hat{f}\left(\frac{(2j-1)\pi 2^m}{2^J}\right) e^{\frac{ik\pi(2j-1)}{2^J}}\right].$$

- Note the need of the characteristic function.

SWIFT: observations

- We can evaluate the density at the extremes of the interval and compute the area underneath the density function as a byproduct, since

$$f_m\left(\frac{h}{2^m}\right) = 2^{\frac{m}{2}} c_{m,k},\ h \in \mathbb{Z}.$$

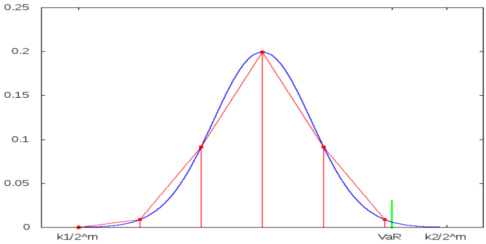
Then

$$\mathcal{A} = \frac{1}{2^{\frac{m}{2}}}\left(\frac{c_{m,k_1}}{2} + \sum_{k_1<k<k_2} c_{m,k} + \frac{c_{m,k_2}}{2}\right).$$

Efficient computation of liquidity-adjusted risk measures

VaR with SWIFT (deterministic Δt)

- We recover the density function of the portfolio change ΔV from its Fourier transform, carrying out the Fourier inversion by means of SWIFT.
- We speed up the computation by using a FFT algorithm.
- We look for the α-quantile of the distribution. To do so:
  1. We find h and $h+1$ such that $2^{\frac{m}{2}}$VaR is located between these two values (it is a sum of trapezoids).



  2. We can accurately compute the VaR using a bisection method within the interval $\left[\frac{h}{2^m}, \frac{h+1}{2^m}\right]$.

ES with SWIFT (deterministic Δt)

- Using Vieta's formula

$$
\begin{aligned}
\text{ES}(\alpha) &= \frac{1}{1-\alpha}\int_{\text{VaR}(\alpha)}^{+\infty} x f(x)\,dx \\
&\approx \frac{1}{1-\alpha}\int_{\text{VaR}(\alpha)}^{b} x \sum_{k=k_1}^{k_2} c_{m,k}\phi_{m,k}(x)\,dx.
\end{aligned}
$$

VaR and ES with SLH (stochastic Δt)

- Let us assume we have the Fourier transform of the deterministic situation: $\hat{f}_{\Delta V}$.
- We assume that the stochastic holding period Δt follows a process with density function $f_{\Delta t}$.
- Making use of the rule $\mathbb{E}\left[\mathbb{E}\left[X \mid Y\right]\right] = \mathbb{E}\left[X\right]$, we have

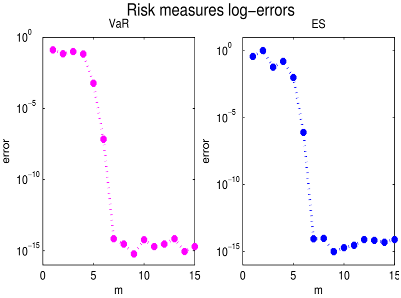
$$\hat{f}_{\Delta V(\Delta t)}(u) = \int_{\mathbb{R}} \hat{f}_{\Delta V}(u)\, f_{\Delta t}(h)\, dh.$$

- Then, using a numerical integration quadrature we compute VaR and ES as in the deterministic situation.

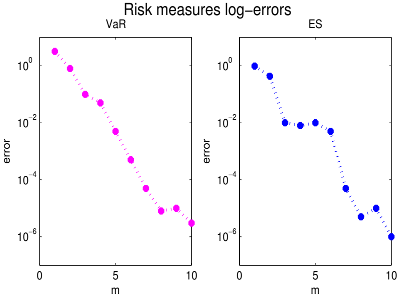
Results: Portfolio dynamics as GBM

- There exists closed form solution.
- The characteristic function of the log-return portfolio change is

$$\hat{f}_{\Delta X_{\Delta t}}(u) = e^{-i\mu u \Delta t - \frac{(\sigma u)^2}{2}\Delta t}.$$



(a) $\Delta t = 1/365$.



(b) $\Delta t \sim exp(10)$.

Results: Under delta-gamma approach(1)

Delta-gamma approximation

It consists of approximate the change in a portfolio value ΔV by

$$\Delta V \approx \Delta V_\gamma := \Theta \Delta t + \delta^T \Delta S + \frac{1}{2} \Delta S^T \Gamma \Delta S,$$

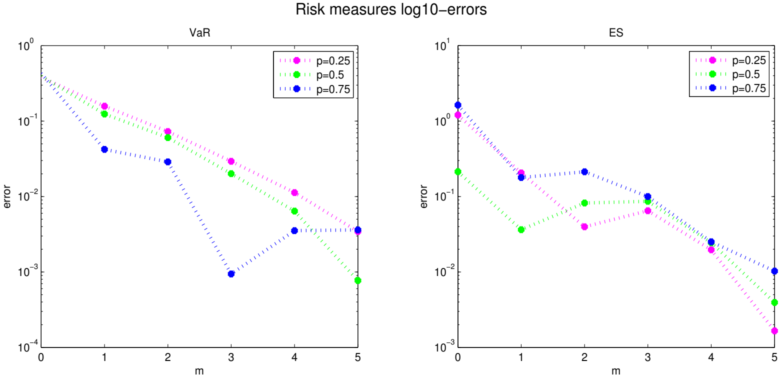
where $S(t) = (S_1(t), \cdots, S_p(t))^T$ are the risk factors, $\Theta = \frac{\partial V}{\partial t}$, $\delta_i = \frac{\partial V}{\partial S_i}$ and $\Gamma_{i,j} = \frac{\partial^2 V}{\partial S_i \partial S_j}$.

- (Mathai and Provost, 1992) It is known the characteristic function of $f_{\widetilde{\Delta V_\gamma}}$ under the assumption that ΔS follows a normal distribution.

Results: Under delta-gamma approach(2)

Table : Bernoulli SLH. Reference prices by Monte Carlo.

Holding Period	Prob - case 1	Prob - case 2	Prob - case 3
10	0.25	0.5	0.75
30	0.75	0.5	0.25
VaR	3.0430	3.0418	3.0364
ES	3.0436	3.0432	3.0414



Conclusions

Conclusions

- SWIFT method has been presented.
- SWIFT method has been used to compute VaR and ES.
- The holding period in VaR and ES has been considered stochastic to reflect the liquidity risk.
- We exhibited the convergence of the method by means of some examples.

Thank you! ☺☺☺