Computational Finance, Numerical Techniques and Applications

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SIAM Student Chapter Day, 23/05/2016

Agenda

- Some basic examples in Financial Engineering
- Option pricing, the mathematical framework
- Numerical techniques in Computational Finance

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- Some basic examples in Financial Engineering
- Option pricing, the mathematical framework
- Numerical techniques in Computational Finance
- Joint work with
  Coen Leentvaar, Xinzheng Huang, Fang Fang, Lech Grzelak, Bin Chen, Bowen Zhang, Marjon Ruijter, Hans van der Weide, Luis Ortiz, Shashi Jain, Alvaro Leitao, Fei Cong, Qian Feng, Ki Wai Chau, Zaza vd Have, Anton vd Stoep, Anastasia Borovykh, Gemma Colldeforns, Andrea Fontanari, Patrik Karlsson, and many others

Application 1

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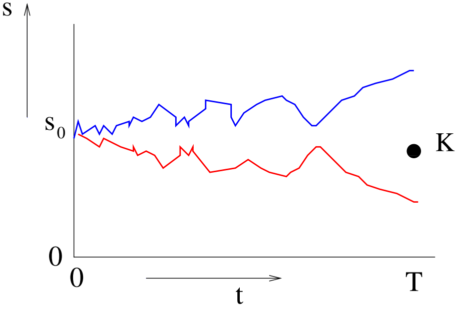
- You can buy insurance against falling stock prices.
- This is the standard put option (i.e. the right to sell stock at a future time point).
- The uncertainty is in the stock prices, but also the counterparty of the contract may go bankrupt!

A put option gives the holder the right to sell in the future at a previously agreed strike price, K.



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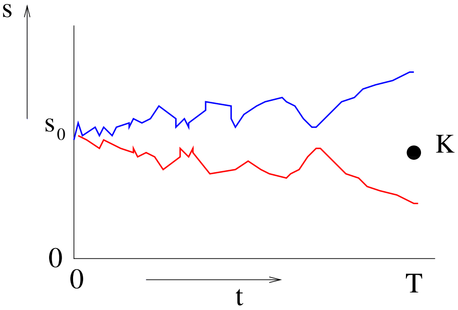

$$v(T,S) = \max(K - S_T, 0) =: h(S_T)$$

$$v(t_0,S) = e^{-r(T-t_0)}\mathbb{E}^Q\{v(S_T,T)|S_0\}$$

Financial engineering, different markets

- Stock market (selling and buying stocks)
- Option exchanges (trading financial options)
- Interest rates market
- FX market
- Credit market
- Commodity market (gold, metals, corn, meat, futures, . . .)
- Energy market (oil, gas, . . .)
- OTC market (over the counter, not traded on a regulated exchange)

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$\Rightarrow$ Each market gives us mathematical questions

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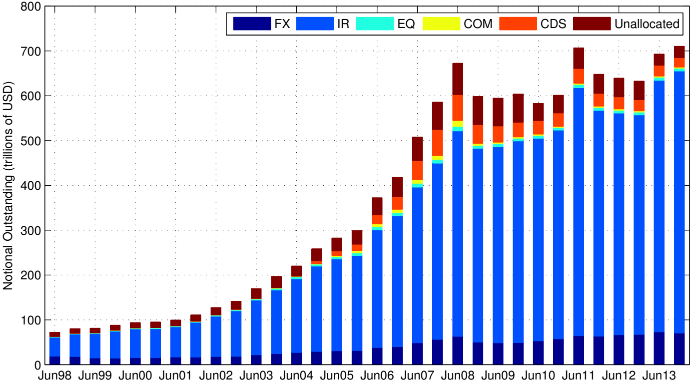
- There is a so-called energy market, where utilities can buy "insurance" (energy derivatives), so that they can buy and sell "power" when the generation is not stable (like swing options).

Market share



Stock options represent a tiny piece of the financial world!

- ⇒ . . . but a good basis for education and research!

Feynman-Kac Theorem (option pricing context)

Given the final condition problem

$$\begin{cases} \frac{\partial v}{\partial t} & + \quad \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0, \\ v(T,S) & = \quad h(S_T) = \text{ given} \end{cases}$$

Then the value, $v(t,S)$, is the unique solution of

$$v(t,S) = e^{-r(T-t)}\mathbb{E}^Q\{v(T,S_T)|\mathcal{F}_t\}$$

with the sum of first derivatives square integrable, and $S = S_t$ satisfies the system of SDEs:

$$dS_t = rS_t dt + \sigma S_t d\omega_t^Q,$$

A pricing approach; European options

$$v(t_0, S_0) = e^{-r(T-t_0)} \mathbb{E}^Q \{ v(T, S_T) | \mathcal{F}_0 \}$$

Quadrature:

$$v(t_0, S_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} v(T, S_T) f(S_T, S_0) dS_T$$

- Trans. PDF, $f(S_T, S_0)$, typically not available, but the characteristic function, $\widehat{f}$, often is.

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- Trans. PDF, $f(S_T, S_0)$, typically not available, but the characteristic function, $\widehat{f}$, often is.
- Similar relations also hold for (multi-D) SDEs and PDEs!

Application 3

- A firm will have some business in America for several years.
- Investment may be in euros, and payment in the local currency.
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- Uncertain processes are the exchange rate, interest rate.
- Banks sell insurance against changing FX rates. The option pays out in the best currency each year.
- There are several risky factors to consider in this case.

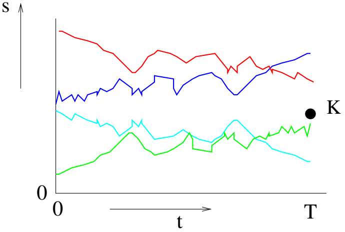
Multi-asset options

Multi-asset options belong to the class of exotic options.

$$v(\mathbf{S}, T) = \max(\max\{S_1, \ldots, S_d\}_T - K, 0) \text{ (max call)}$$

$$v(\mathbf{S}, t_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} v(\mathbf{S}, T) f(\mathbf{S}_T|\mathbf{S}_0) d\mathbf{S}$$

- $\Rightarrow$ High-dimensional integral or a high-D PDE.



Increasing dimensions: Multi-asset options

- The problem dimension increases if the option depends on more than one asset S_i (multiple sources of uncertainty).
- If each underlying follows a geometric (lognormal) diffusion process,
- Each additional asset is represented by an extra dimension in the problem:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d}[\sigma_i\sigma_j\rho_{i,j}S_iS_j\frac{\partial^2 v}{\partial S_i \partial S_j}] + \sum_{i=1}^{d}[rS_i\frac{\partial v}{\partial S_i}] - rv = 0 \ .$$

- Required information is the volatility of each asset σ_i and the correlation between each pair of assets $\rho_{i,j}$.

Numerical Pricing Approach

- One can apply several numerical techniques to calculate the option price:
  - Numerical integration,
  - Monte Carlo simulation,
  - Numerical solution of the partial-(integro) differential equation
- Each of these methods has its merits and demerits.
- Numerical challenges:
  - The problem's dimensionality
  - Speed of solution methods
  - Early exercise feature ($\rightarrow$ free boundary problem)

SABR model, FX market

- SABR model (Hagan 2002) is an established model for foreign-exchange (FX) modeling.
- A parametric local volatility component in terms of β,

$$\begin{aligned} dS_t &= \sigma_t S_t^{\beta} d\omega_S, \\ d\sigma_t &= \alpha \sigma_t d\omega_\sigma. \end{aligned}$$

S_t the forward price, r interest rate, T the maturity. σ_t is stochastic volatility, ω_S, ω_σ are correlated BMs (i.e. $\omega_S \omega_\sigma = \rho t$).

- Corresponding PDE for the valuation of FX options:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 v}{\partial S^2} + \rho\alpha S^{\beta}\sigma^2 \frac{\partial^2 v}{\partial\sigma\partial S} + \frac{1}{2}\alpha^2\sigma^2\frac{\partial^2 v}{\partial\sigma^2} - rv = 0$$

Financial engineering; Work at Banks

- Financial engineering, pricing approach:
  1. Start with some financial product
  2. Model asset prices involved (SDEs)
  3. Calibrate the model to market data (Numerics, Opt.)
  4. Model product price correspondingly (PDE, Integral)
  5. Price the product of interest (Numerics, MC)

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  5a. Price the risk related to default (SDE, Opt.)

  6. Understand and remove risk (Stoch., Opt., Numer.)

Academic vs industrial research

- Our academic research is inspired by financial applications
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- Aim of academic research: develop novel insights and algorithms (not necessarily directly used), understand the results, consider errors made, publish papers.
- Search compromise between a fix that works and a fundamental solution which is slow.

Our work

- We work in numerical analysis and scientific computing;
- Application area is financial engineering; topic is computational finance;
- Financial applications are governed by PDEs with different features and properties than in other engineering areas.

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- Challenge is the interaction between probability theory (stochastic processes) and numerical mathematics (efficient computation of expectations, quantiles, etc.)
- Reducing uncertainty; grip on complexity

Fourier-Cosine Expansions, COS Method (with Fang, Bowen, Marjon, Gemma, Anastasia, Luis)

- The COS method:
  - based on the availability of a characteristic function.
  - Replace the density by its Fourier-cosine series expansion;
  - Coefficients have simple relation to characteristic function.
  - Exponential convergence;
  - Sensitivities obtained at no additional cost.

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- Academic aim: Increase the range of applicability of the method.
- Towards wavelets (understand convergence)

Results, Heston stochastic volatility PDE

$$\frac{\partial v}{\partial t} = \frac{1}{2}S^2 y \frac{\partial^2 v}{\partial S^2} + \rho\gamma S y \frac{\partial^2 v}{\partial S \partial y} + \frac{1}{2}\gamma^2 y \frac{\partial v}{\partial y^2} + rS\frac{\partial v}{\partial S} + \kappa(\overline{\sigma} - y)\frac{\partial v}{\partial y} - rv.$$

- GPU computing: Multiple strikes for parallelism, 21 IC's.

Heston model				
	N	64	128	256
MATLAB	msec	3.850890	7.703350	15.556240
	max.abs.err	6.0991e-04	2.7601e-08	$< 10^{-14}$
GPU	msec	0.177860	0.209093	0.333786

Table 1: Maximum absolute error when pricing a vector of 21 strikes.

- Exponential convergence, Error analysis in our papers.
- Also works with Shannon wavelets (SWIFT) instead of cosines.

- The semilinear partial differential equation:

$$\frac{\partial v(t,x)}{\partial t} + \mathcal{L}v(t,x) + g(t,x,v,\sigma(x)Dv(t,x)) = 0,\ v(T,x) = h(x),$$

We can solve this PDE by means of the FSDE:

$$dX_s = \mu(X_s)ds + \sigma(X_s)d\omega_s,\ X_t = x.$$

and the BSDE:

$$dY_s = -g(s, X_s, Y_s, Z_s)ds + Z_s d\omega_s,\ Y_T = h(X_T).$$

- Theorem:

$$Y_t = v(t, X_t),\ Z_t = \sigma(X_t)Dv(t, X_t).$$

is the solution to the decoupled FBSDE.

⇒ Use COS method or SWIFT to solve BSDEs.

Application 4

- Consider a pension fund with incoming money and obligations to pay pensions in the future.
- When the regulations regarding the estimation of the future earnings change and the ratio between assets (payments, investments) and liabilities (pensions) change, how to assure that future pensions can be paid?
- Can we quantify the risk that pensions cannot be paid out?

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- The uncertainty is in the asset-liability ratio, the interest rate, ...
- So-called embedded options exist, that become active when the asset/liability ratio is out of order (ALM).
- What is the optimal portfolio for the fund, over time, in stocks, bonds real estate and/or options?

HJB equation, dynamic programming

- Suppose we consider the Hamilton-Jacobi-Bellman (HJB) equation:

$$
\begin{aligned}
\frac{\partial v(t,x)}{\partial t} &+ \sup_{a \in A} \{ \mu'(x,a) D v(t,x) + \frac{1}{2} \mathrm{Tr}[D^2 v(t,x) \sigma \sigma'(x,a)] \\
&+ g(t,x,a) \} = 0, \\
v(T,x) &= h(x).
\end{aligned}
$$

It is associated to a stochastic control problem with value function

$$
v(t,x) = \sup_{\alpha} \mathbb{E}_t^x \left[\int_t^T g(s, X_s^\alpha, \alpha_s) ds + h(X_T^\alpha) \right],
$$

where X_s is the solution to the controlled FSDE

$$
dX_s^\alpha = \mu(X_s^\alpha, \alpha_s) ds + \sigma(X_s^\alpha, \alpha_s) d\omega_s, \; X_t^\alpha = x.
$$

Dynamic Mean-Variance Optimization, with Fei

- Target function:

$$\max_{\{\alpha_t\}_{t=0}^{T-\Delta t}} \Big[\mathbb{E}[W_T|W_0] - \lambda \cdot \mathrm{Var}[W_T|W_0]\Big], \tag{1}$$

where $\{\alpha_t\}_{t=0}^{T}$ are asset allocations at sequential time steps and the wealth evolves following:

$$W_{t+\Delta t} = W_t \cdot (\alpha_t \cdot R_t^e + R_f), t = 0, \ldots, T - \Delta t.$$

- Constraints on the asset allocations
- R_t^e: excess return of the risky asset R_f: return of the risk-free asset
- General: Using Bellman's dynamic programming principle
- Not immediate for the mean-variance case! Because:

$$\mathrm{Var}[\mathrm{Var}[W_T|\mathcal{F}_t]|\mathcal{F}_s] \neq \mathrm{Var}[W_T|\mathcal{F}_s], s \leq t.$$

Application 5

- Two businesses, one established and one trying to enter the market. Businesses are represented by their value, S_1, S_2.
- Two stochastic processes, S_1, S_2, with time-dependent $\mu(t)$ and $\sigma(t)$, maybe negatively corrected, where
  - S_1: is a process with decreasing μ (initially high) and increasing σ (initially relatively low). "A certain business with a good margin initially, but when a competitor enters the market it is future profit is uncertain."
  - S_2: A process (profit margin) with high initial costs, initially low in terms of $\mu(t)$ and high in terms of $\sigma(t)$, because it is an uncertain participant with a new product that may catch up. $\mu(t)$ may increase in the future, and then $\sigma(t)$ will decrease.
- What is the market share over time of the market participants?

Optimal stopping, dynamic programming

- Control over the contract's terminal time.

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\omega_t,$$

- With $t \in [0, T]$, and stopping times $\mathcal{T}_{t,T}$, the finite horizon optimal stopping problem is formulated as

$$v(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[\int_t^\tau e^{-r(T-t)} g(s, X_s)ds + e^{-r(\tau - t)} h(X_\tau)\right].$$

- The value function v is related to the HJB variational inequality:

$$\min[-\frac{\partial v}{\partial t} - \mathcal{L}v - g, v - h] = 0,$$

- The problem is called a free boundary problem.
- C is the continuation region, the complement set is the stopping or exercise region (receive the reward h).

- Backward Dynamic Programming:

- COS and SWIFT work fine up to 3 dimensions.
  Higher D: Monte Carlo: Stochastic Grid Bundling Method, SGBM;
- The Bermudan option at time t_m and state $\mathbf{S}_{t_m}$ is given by

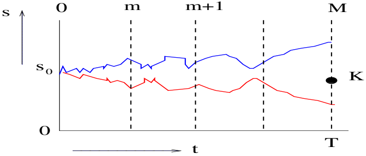
$$v_{t_m}(\mathbf{S}_{t_m}) = \max(h(\mathbf{S}_{t_m}), c_{t_m}(\mathbf{S}_{t_m})). \tag{2}$$

- The continuation value c_{t_m}, is :

$$c_{t_m}(\mathbf{S}_{t_m}) = e^{-r\Delta t_m}\mathbb{E}\left[v_{t_{m+1}}(\mathbf{S}_{t_{m+1}})|\mathbf{S}_{t_m}\right].$$

Towards higher dimensions: with Shashi, Fei, Alvaro, Qian

- Backward Dynamic Programming:



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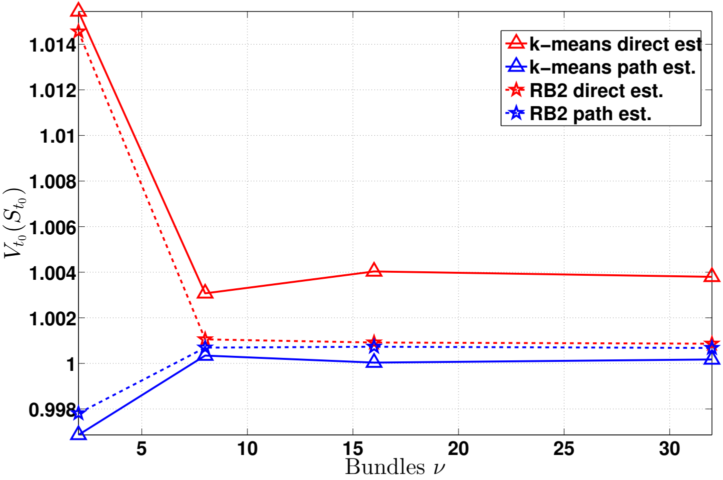

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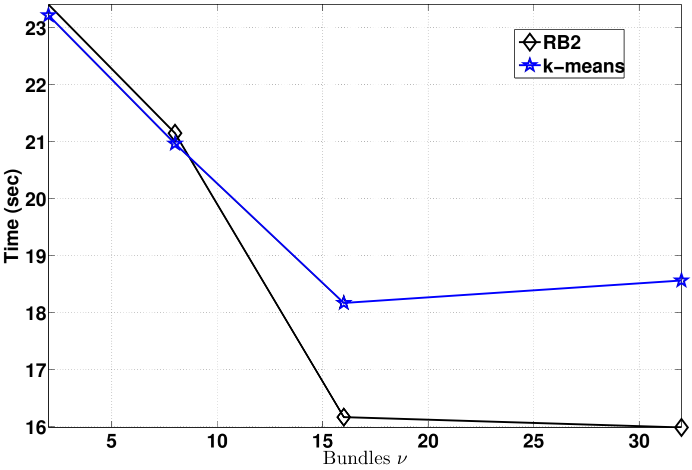
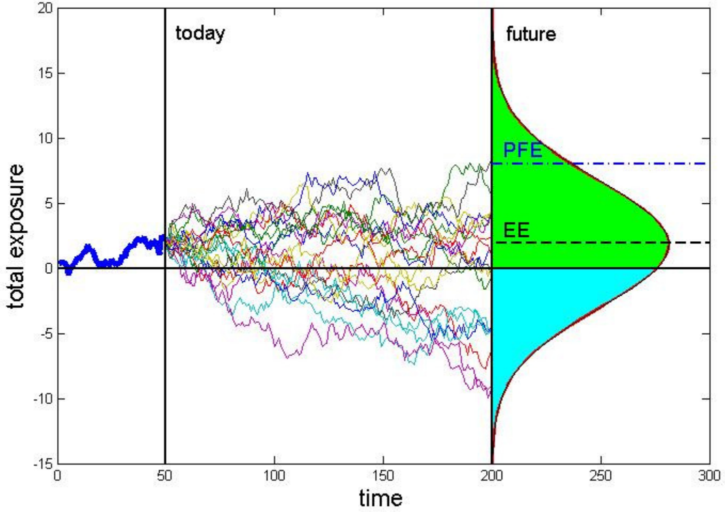


Expected exposure in CVA (joint work with Qian, Kees, Drona, Shashi, Patrik)



Application 6

- Consider an oil company, and you developed a way to get oil out cheaper.
- Will the company invest in R&D related to your invention? Will they adopt your new technique?
- This may partly depend on the uncertain oil price.
- When the oil price is high, it may be worthwhile to invest in new technology, as the rate of return may be favorable.
- When the oil price is low, profit depends on number of employees ...
- This case can be analyzed, to some extent, with financial models and computations (real option evaluation).

Scientific computing aspect

- Bowen (PhD 2013), COS method on GPUs (calibration)
- Hans Knibbe (PhD 2015), 3D PDEs (from seismics) on GPUs
- Alvaro (PhD 2017), Monte Carlo methods, CVA on GPUs

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Little-Green Machine

20 general computing nodes

- 2 Intel quadcore E5620
- 24 GB RAM
- 2 TB disk
- 2 NVIDIA GTX480

Founded by

- University of Leiden
- NWO
- TU Delft
- KNMI

Observations, Computational Finance

- Many requests for quantitative answers these days, also from consulting companies
- A focus on risk management questions
- Industrial requirements, many constraints etc.
- Let's not forget that it is only a mathematical model!

$\Rightarrow$ Challenge: Incorporate (big) financial data

Computational finance

- Efficient valuation of financial options
  → COS method efficient for a variety of options (barrier, Asian, multi-asset)
  - wavelets SWIFT, with Luis Ortiz Gracia
  - SGBM: High-D American options, Monte Carlo simulation
  - Lech Grzelak: Interpolation, Stochastic Collocation Monte Carlo methods (SCMC sampler)
- Risk management
  - Accurate hedge parameters;
  - Numerical estimation of tail probabilities and Expected Shortfall, Value-at-Risk;
  - (Counterparty) Credit risk and other types of risk;
- Portfolio optimization
  - Energy portfolio, real options analysis
  - Dynamic portfolios for pensions (target based vs time-consistent mean-variance strategy)