Induced Dimension Reduction method to solve the Quadratic Eigenvalue Problem

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Student Krylov Day, February 2nd, 2015

How to solve **this problem?**

Source

- www.youtube.com. Millennium Bridge Opening Day

Problem

Motivation

Vibration Analysis of Structural Systems

The Quadratic eigenvalue problem: Given the matrices M, D, and K of dimension n, find the scalar λ and the vector $\mathbf{x}$, such that:

$$(\lambda^2 M + \lambda D + K)\mathbf{x} = 0$$

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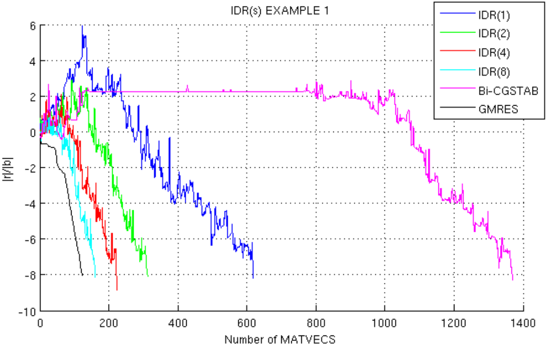
IDR(s) method for solving system of linear equation



References

- Sonneveld, and M. B. van Gijzen, SIAM J. Sci. Computing 31, 1035–1062 (2008).

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IDR(s) method for solving system of linear equation

IDR(s) Theorem

$$\mathcal{G}_0 \equiv \mathbb{C}^n$$

$$\mathcal{G}_{j+1} \equiv (A - \mu_{j+1} I)(\mathcal{G}_j \cap \mathcal{S}) \quad j = 0, 1, 2 \ldots,$$

1. $\mathcal{G}_{j+1} \subset \mathcal{G}_j$, and
2. dimension($\mathcal{G}_{j+1}$) $<$ dimension($\mathcal{G}_j$) unless $\mathcal{G}_j = \{\mathbf{0}\}$.

References

- Sonneveld, and M. B. van Gijzen, SIAM J. Sci. Computing 31, 1035–1062 (2008).

IDR(s) method for solving system of linear equation

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- http://www.panoramio.com user: Jetzabel.

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A Hessenberg decomposition based on IDR(s)

Create $\mathbf{w}_{k+1} \in \mathcal{G}_{j+1} \equiv (A - \mu_{j+1} I)(\mathcal{G}_j \cap \mathcal{S})$

Assume that $\mathbf{w}_k, \mathbf{w}_{k-1}, \ldots, \mathbf{w}_{k-(s+1)}$ are in $\mathcal{G}_j$ and $\mathcal{S} \equiv P^\perp$.

$$\mathbf{w}_{k+1} = (A - \mu_{j+1} I)(\mathbf{w}_k - \sum_{i=1}^{s} \beta_i \mathbf{w}_{k-i})$$

with $P^T(\mathbf{w}_k - \sum_{i=1}^{s} \beta_i \mathbf{w}_{k-i}) = \mathbf{0}$, then

$$A\mathbf{w}_k = \mathbf{w}_{k+1} + \mu_{j+1}\mathbf{w}_k + A\sum_{i=1}^{s} \beta_i \mathbf{w}_{k-i} - \mu_{j+1}\sum_{i=1}^{s} \beta_i \mathbf{w}_{k-i}$$

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A Hessenberg decomposition based on IDR(s)

$$AW_m = W_m H_m + \mathbf{f} \mathbf{e}_m^T$$

- $\Lambda(H_m)$ approximated some eigenvalues of A
- Short-term recurrence method.
- Implicit filter polynomial.
- Low-memory requirements.

References

- R. Astudillo and M. B. van Gijzen, AIP Conf. Proc. 1558, 2277 (2013).
- -, Technical Report 14-04, Delft University of Technology, The Netherlands, 2014.

Solving the Quadratic eigenvalue problem with IDR(s)

Linearization and properties

The problem

$$(\lambda^2 M + \lambda D + K)\mathbf{x} = 0$$

can be transformed into the Generalized Eigenvalue Problem

$$C\mathbf{y} = \lambda G\mathbf{y},$$

where

$$C = \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}$$

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Solving the Quadratic eigenvalue problem with IDR(s)

Linearization and properties

$$S = G^{-1}C = \begin{bmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{bmatrix}$$

if $\mathbf{v} = [\mathbf{r}_0, \mathbf{0}]^T$, then

$$\begin{bmatrix} \mathbf{r}_j \\ \mathbf{r}_{j-1} \end{bmatrix} = S^j \mathbf{v}$$

References

- Z. Bai and Y. Su, SIAM J. Matrix Anal. Appl. 26, pp. 640–659 (2005).

Solving the Quadratic eigenvalue problem with IDR(s)

Linearization and properties

$$SW_m = W_m H_m + \mathbf{f}\mathbf{e}_m^T$$

if

$$W_m = \begin{bmatrix} W_m^{(t)} \\ W_m^{(b)} \end{bmatrix}$$

$$\begin{aligned} (-M^{-1}D)W_m^{(t)} + (-M^{-1}K)W_m^{(b)} &= W_m^{(t)} H_m + \mathbf{f}^{(t)}\mathbf{e}_m^T \\ W_m^{(t)} &= W_m^{(b)} H_m + \mathbf{f}^{(b)}\mathbf{e}_m^T \end{aligned}$$

Then

$$W_m^{(b)} = W_m^{(u)} T,$$

Where T is upper triangular, then we save only $W_m^{(u)}$.

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A Hessenberg decomposition based on IDR(s)
for QEP

$$\begin{bmatrix} -M^{-1}D & -M^{-1}K \\ I & 0 \end{bmatrix} W_m = W_m H_m + \mathbf{f}\mathbf{e}_m^T$$

- $\Lambda(H_m)$ approximated some eigenvalues of QEP:

$$(\lambda^2 M + \lambda D + K)\mathbf{x} = 0$$

- Short-term recurrence method.
- Implicit filter polynomial.
- Low-memory requirements.
- Operation $\mathcal{O}(n)$

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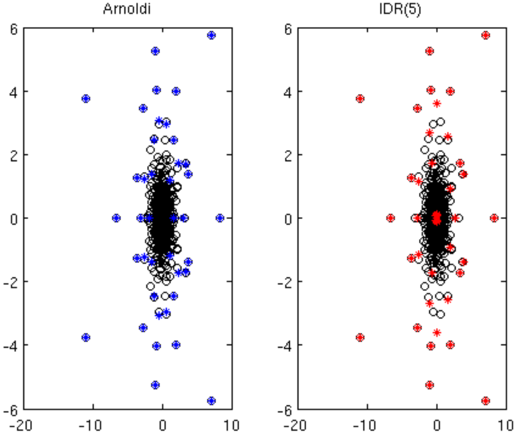
Numerical tests: n = 200; M = rand(n); D = rand(n); K = rand(n);



Arnoldi: 0.041 secs. IDR(s): 0.025 secs.

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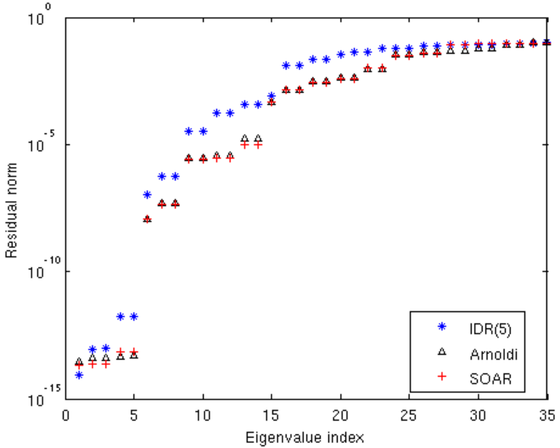
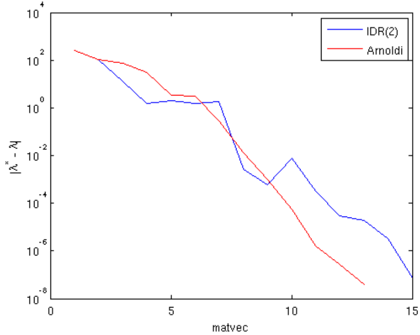


Arnoldi: 0.041 secs. IDR(s): 0.025 secs.

Numerical tests: Wave propagation in a room

Dimension: 1681



Convergence to the known eigenvalue $\lambda^* = -5 + 216i$
Arnoldi: 0.51 secs. IDR(s): 0.42 secs.

References

- Sleijpen, G. L. G.; van der Vorst, H. A. and van Gijzen, M. B., SIAM News 29, pp 8-19, (1996).

Thank you for your attention. Questions, comments, suggestions are welcome.