UMC Utrecht

Mathematical Modelling in Magnetic Resonance Imaging

Alessandro Sbrizzi

Netherlands Organisation for Scientific Research

VENI Grant 15115

University Medical Center Utrecht

1973

Daily Mail

MONDAY, JANUARY 1, 1973

3p

A HAPPY NEW YEAR TO YOU ALL

Daily Mail COMMENT

'For ten years the Mail has campaigned for this day. We have not wavered in our conviction that Britain's best and brightest future is with Europe.'

EUROPE, HERE WE COME!

WE welcome this day.

All bygone adventures to unite the peoples of our Continent have been empires of the sword that have been sundered by the sword.

This European Community of which we are now a part is different.

It is a free association of nations drawn together by a common will to bury the sword.

To transform the cockpit of Europe into a power for peace—that was the ideal that took seed amid the rubble more than a quarter of a century ago.

Had we had a hand in it then, the Europe we now belong to would be more to our democratic taste and we would not have to suffer the full economic absurdities of the Common Agricultural Policy.

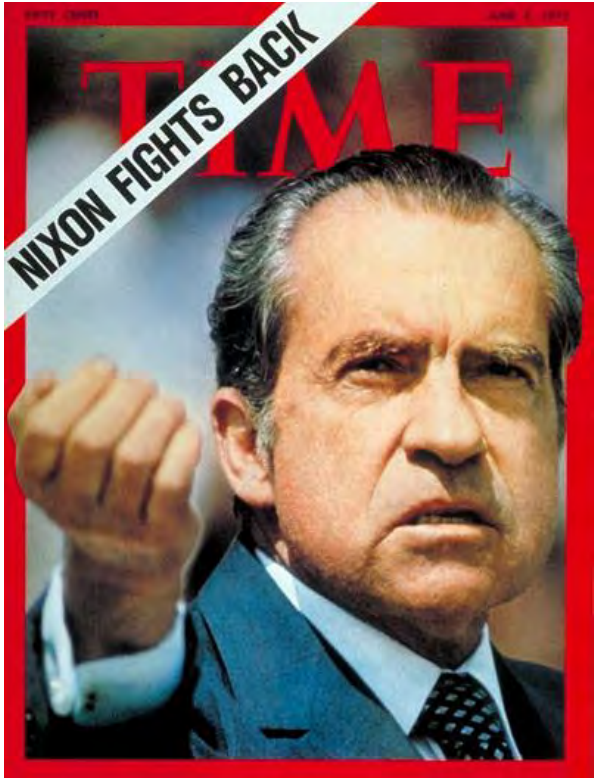
But these are still early days, pioneering days.

The shape of European unity that will eventually rise on these foundations has not been determined.

Today our destiny as a nation has committed us to mix the mortar and help build this new Europe, to ensure that the poorer regions in Britain and the other countries share in the prosperity to come.

For ten years the Daily Mail has campaigned for this commitment. We have not wavered in our conviction that Britain's best and brightest future is

THE DAILY MAIL EUROPRIZE



Nature, 1973, pag. 190-191

Image Formation by Induced Local Interactions: Examples Employing Nuclear Magnetic Resonance

P. C. LAUTERBUR

Department of Chemistry,
State University of New York at Stony Brook,
Stony Brook, New York 11790

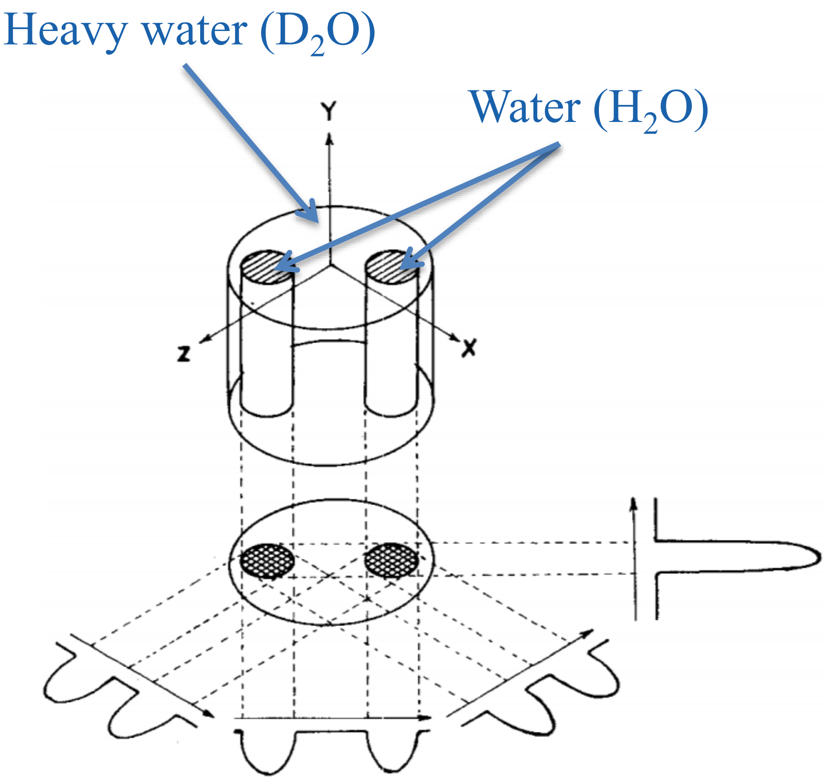


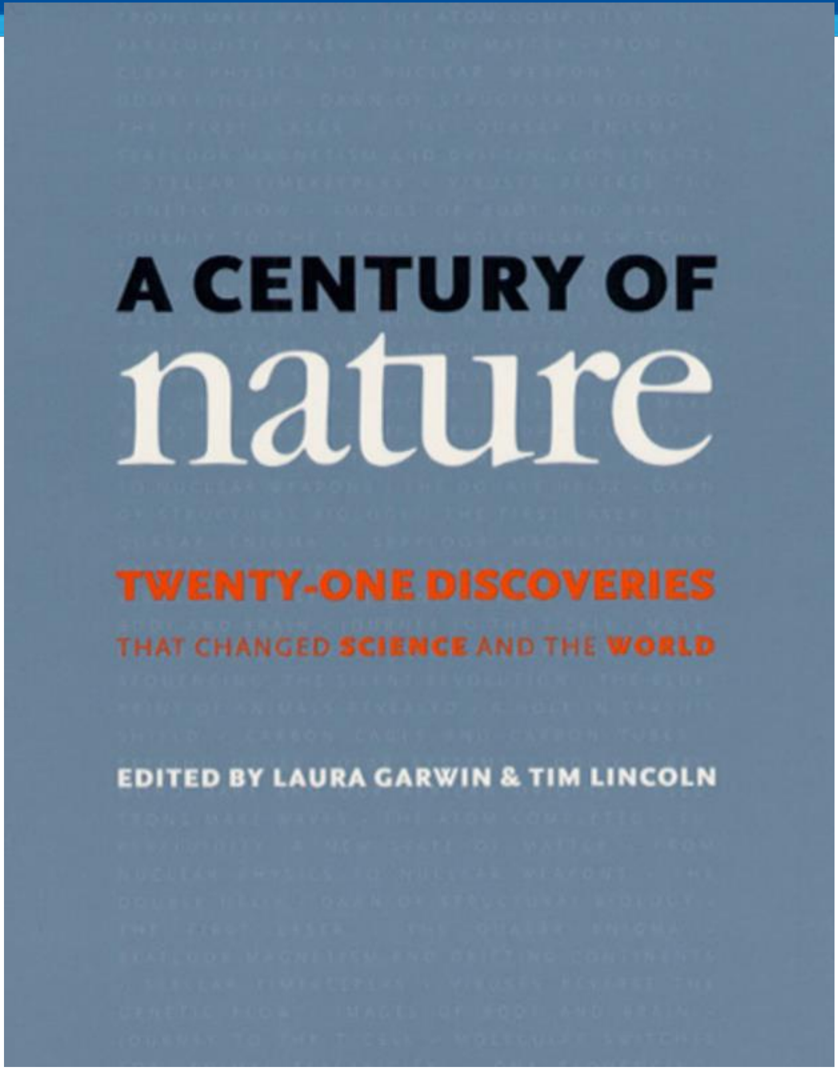
Fig. 1 Relationship between a three-dimensional object, its two-dimensional projection along the Y-axis, and four one-dimensional projections at 45° intervals in the XZ-plane. The arrows indicate the gradient directions.

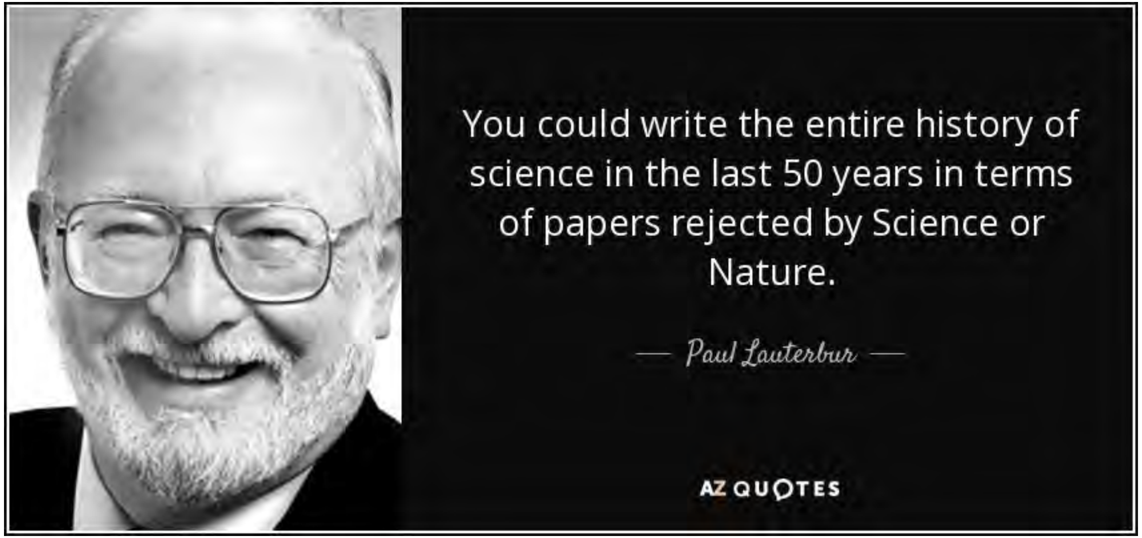
" the signal in the presence of a field gradient represents a **one-dimensional projection** of the H_2O content of the object, **integrated over planes** perpendicular to the gradient direction, as a function of the gradient coordinate…."

"… **combine several projections**, … using one of the available methods for **reconstruction of objects from their projections**."

“**The variations in water contents and proton relaxation times among biological tissues** should permit the generation (…) of useful zeugmatographic images (…). A possible application of considerable interest at this time would be to the **in vivo study** of malignant tumours …”

Fig. 2 Proton nuclear magnetic resonance zeugmatogram of the object described in the text, using four relative orientations of object and gradients as diagrammed in Fig. 1.


A CENTURY OF
nature
TWENTY-ONE DISCOVERIES
THAT CHANGED SCIENCE AND THE WORLD
EDITED BY LAURA GARWIN & TIM LINCOLN


You could write the entire history of science in the last 50 years in terms of papers rejected by Science or Nature.
— Paul Lauterbur —
AZ QUOTES

1975: Enters Fourier transform

NMR Fourier Zeugmatography

ANIL KUMAR, DIETER WELTI, AND RICHARD R. ERNST

Laboratorium für Physikalische Chemie, Eidgenössische Technische Hochschule, 8006 Zürich, Switzerland

"(…) The spatial spin density function can then be reconstructed by a straightforward **two- or three-dimensional Fourier transformation**. One of the important features of this method is the **homogenous error distribution over the entire frequency range** such that low and high frequency components can be reconstructed with equal accuracy. The method can easily be implemented on a **small on-line computer**."

The early ’80s: shaping the *k*-space

Spin warp NMR imaging and applications to human whole-body imaging

W A Edelstein, J M S Hutchison, G Johnson and T Redpath

Physics in Medicine and Biology, Volume 25, Number 4

1980

“In the spin warp technique**, all the projections are along the same direction,** and therefore inhomogeneity will manifest itself only as a geometric distortion in the final image; there will be **no smearing of imaging information**.”

FFT for image reconstruction

The early ’80s: shaping the *k*-space

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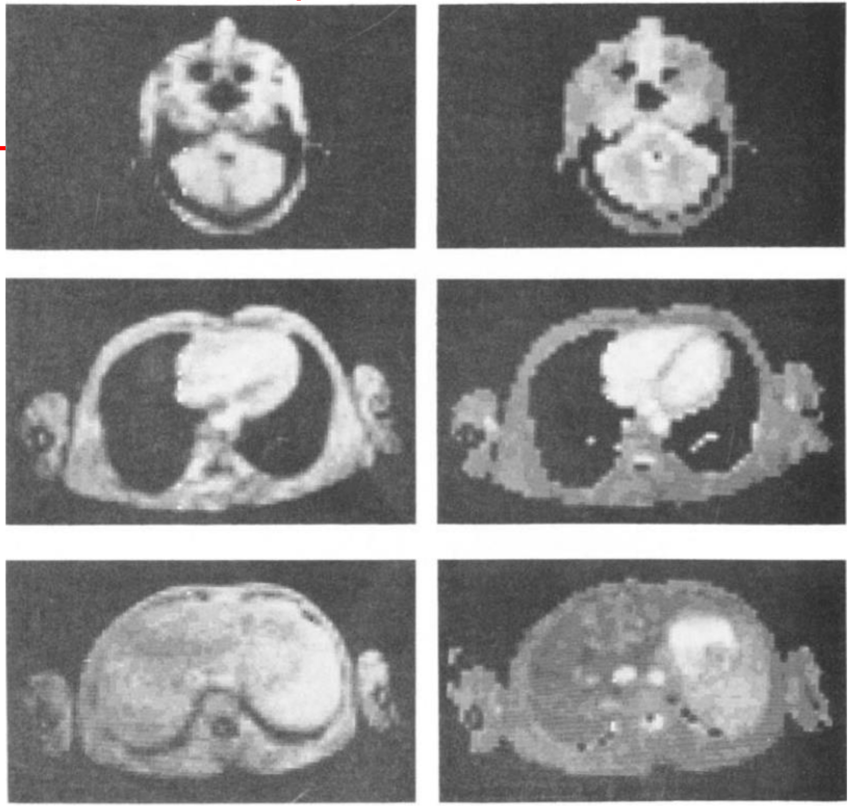

64 x 64 FFT

The early '80s: shaping the *k*-space

A Simple Graphical Representation of Fourier-Based Imaging Methods

STIG LJUNGGREN

1983

Department of Physical Chemistry, The Royal Institute of Technology, S-100 44, Stockholm 70, Sweden

$$S(t) = \int \rho(\mathbf{r}) \exp\left(i\gamma\mathbf{r} \cdot \int_0^t \mathbf{G}(t')dt' \right) d\mathbf{r}$$

$$\mathbf{k}(t) = \gamma \int_0^t \mathbf{G}(t')dt'$$

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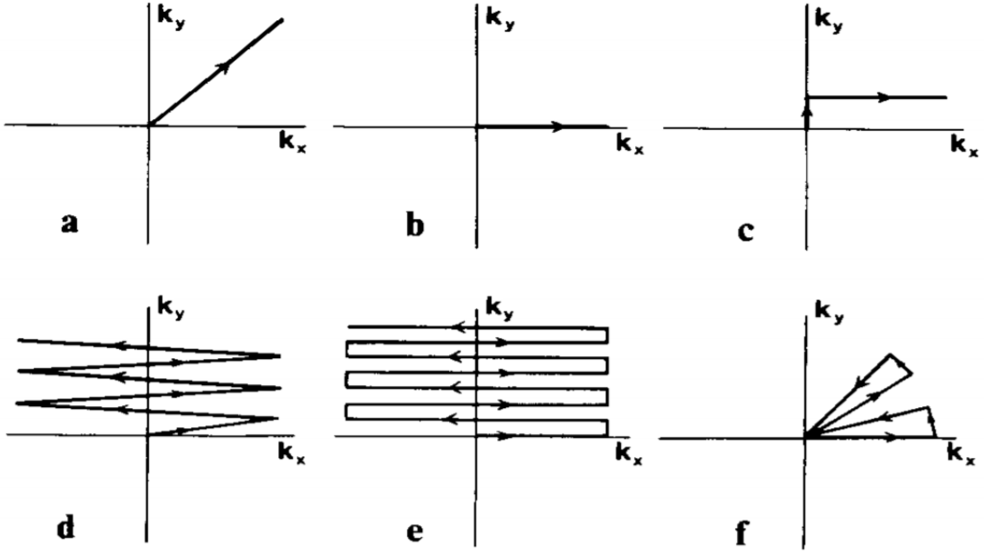


FIG. 1. Scanning patterns in the **k** plane: (a) projection reconstruction method; (b) line scan method; (c) Fourier imaging; (d) echo planar imaging method; (e) modified EPI method; (f) modified projection reconstruction method.

Ljunggren, 1983

The early ’80s: shaping the *k*-space

“ …, it is believed that many unnecessary mathematical complications such as the use of the so-called **projection functions may be avoided** in this way”

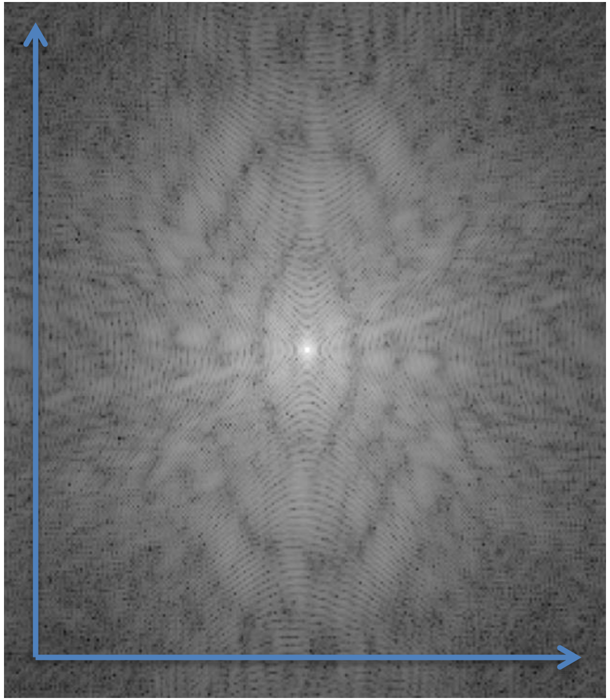
“The rather loosely outlined applications suggested above **should not be taken too seriously**.”

Ljunggren, 1983

MRI by 1983

K-space

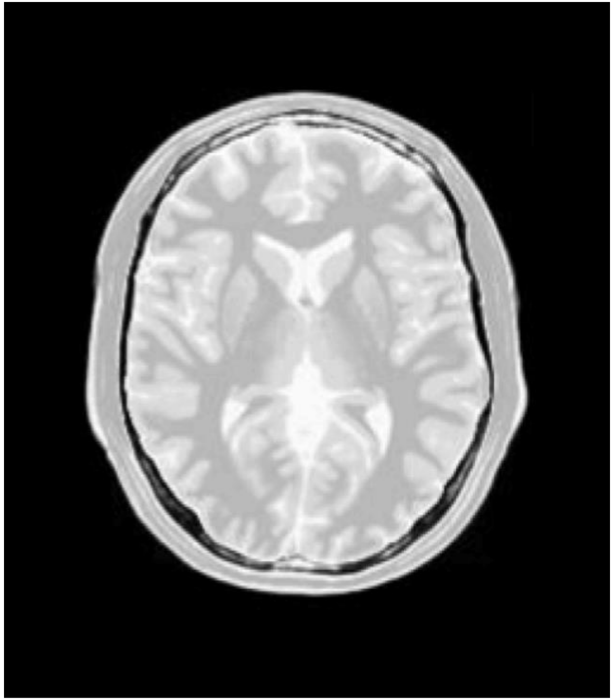
k_y



k_x

Image space

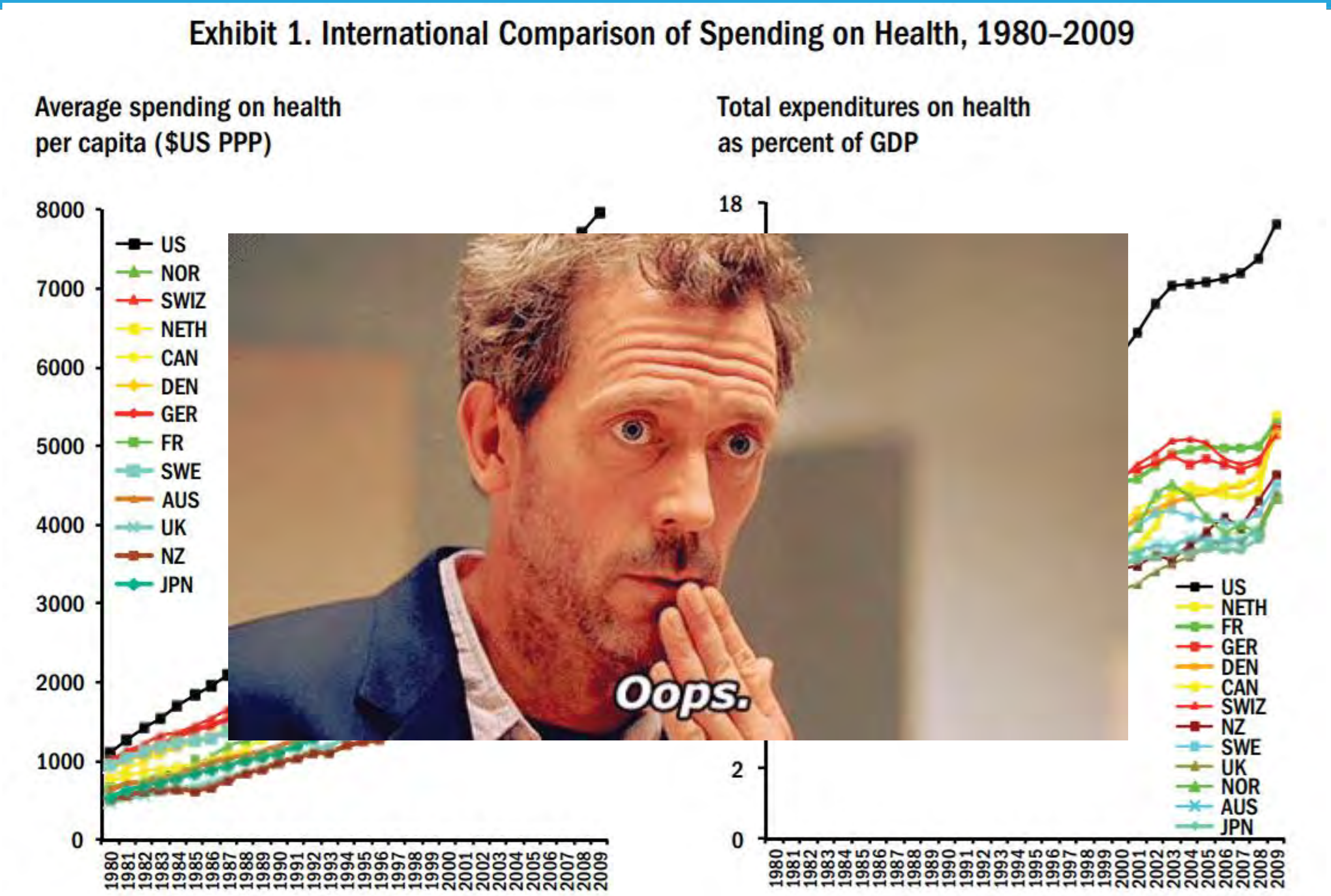
y



x

MRI through the ’90s

- MRI becomes a mature/robust technique
- Numbers of clinical scanners (and scans) steadily increase in US and Europe



D.A. Squires, The Commonwealth Fund, 2012

Making MRI cheaper = faster

Parallel Imaging (From 1997)

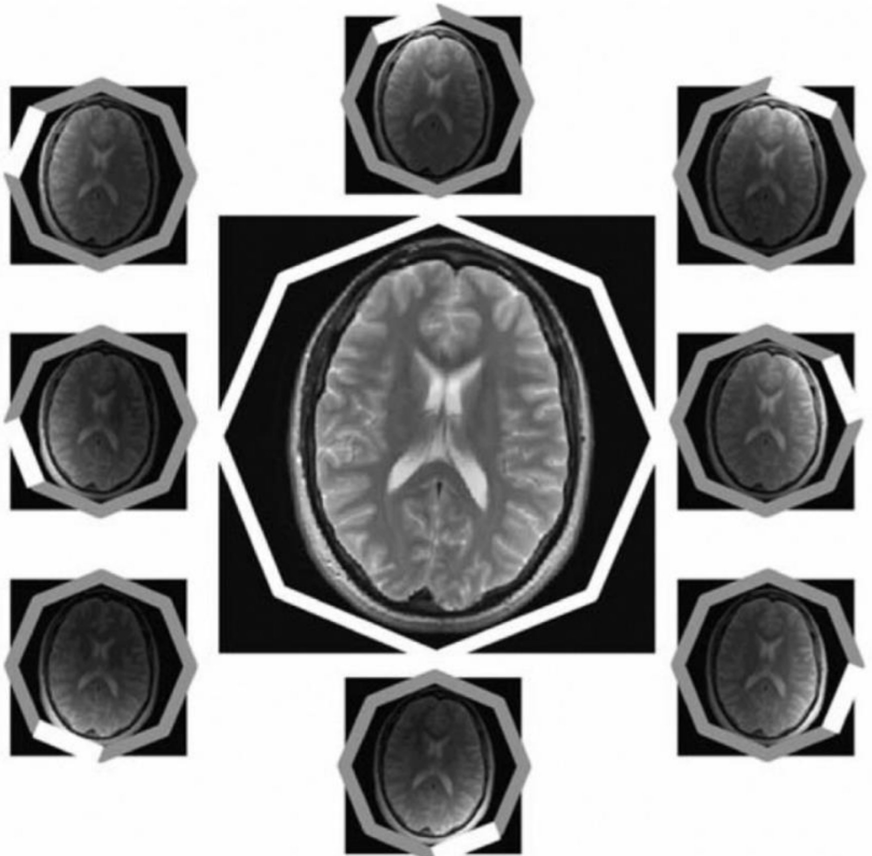


Image from *Deshmane et al JMRI 2012*

Employ several receiver antennas, **simultaneously**

Individual **additional** spatial information

Undersample the *k*-space

Parallel Imaging (From 1997)

$$d_{\gamma,\kappa} = \int M(\boldsymbol{r})\, \mathrm{enc}_{\gamma,\kappa}(\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$$

$$\mathrm{enc}_{\gamma,\kappa}(\boldsymbol{r}) = s_{\gamma}(\boldsymbol{r}) e^{j\boldsymbol{k}_{\kappa}\boldsymbol{r}}$$

Coil dependent spatial weighting

Parallel Imaging (From 1997)

$$\mathrm{enc}_{\gamma,\kappa}(\boldsymbol{r}) = s_\gamma(\boldsymbol{r})e^{jk_\kappa \boldsymbol{r}}$$

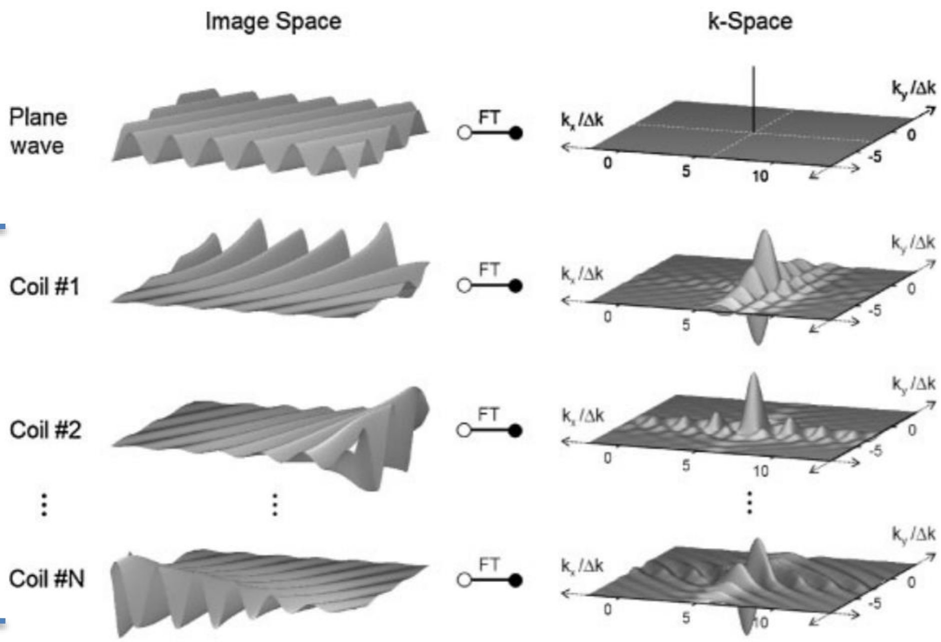


Image from *Prussmann NMR Biomed 2006*

Parallel Imaging (From 1997)

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Parallel imaging leads to (large scale) least-squares reconstruction

→ iterative methods (CG, LSQR)

→ preconditioning

→ Tikhonov regularization

The magic of compressed sensing

- Since 1970s in several applied fields : exploit **sparse representation of data** through ℓ_1 norm for regularization of ill-conditioned/underdetermined systems

- Beginning 2000s, California: Candes (Stanford), Romberg (Caltech/UCLA), Tao (UCLA), Donoho (Stanford). Theoretical foundations are laid. Correct recovery from highly undersampled data (!). Concept of **incoherence/randomization.**

- 2004: *Convex Optimization* book by Boyd & vandenBerghe (more than 35,000 citations by May 2017). **Tractable algorithms**.

The Nobel Prize in Physiology or Medicine 2003

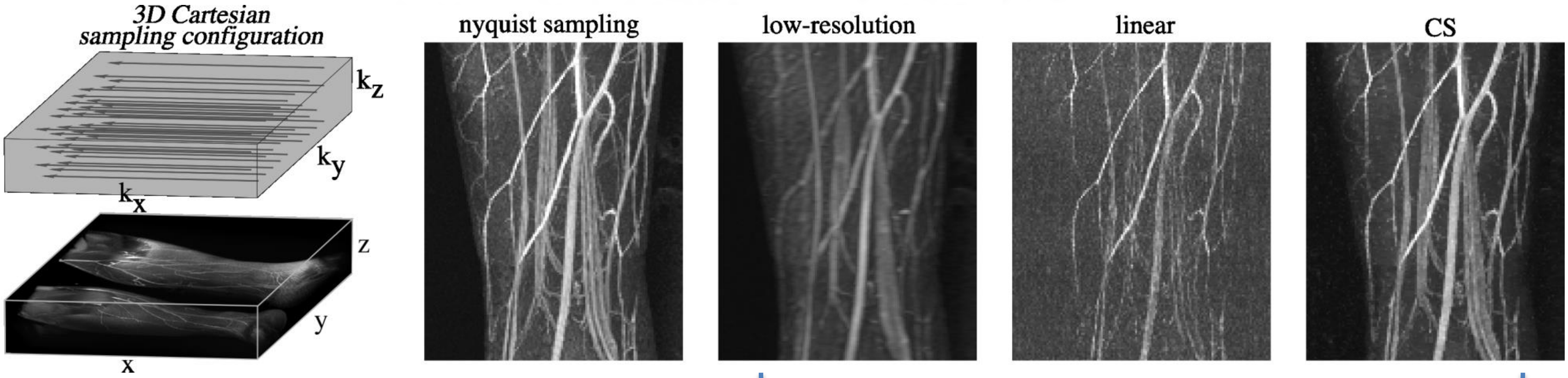
Paul C. Lauterbur
Prize share: 1/2

Sir Peter Mansfield
Prize share: 1/2

Sparse MRI: The Application of Compressed Sensing for Rapid MR Imaging

2007

Michael Lustig,[1*] David Donoho,[2] and John M. Pauly[1]



(10-fold undersampling)

Compressed sensing in MRI

- Development of ℓ_2/ℓ_1 **minimization** algorithms

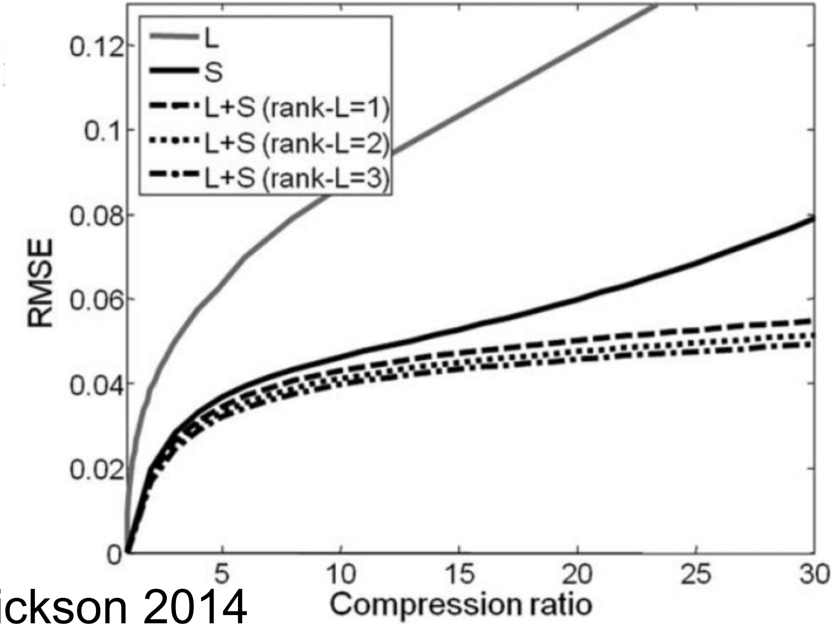
- Optimal **undersampling** schemes?

- Optimal **sparse representation**?

- CS-MRI is now a **clinical product** from major vendors (<u>modern math enters the clinic!</u>)

Recent developments in CS MRI

- Recover image, M, such that $M = L + S$
- L : low-rank matrix (temporally correlated **background**)
- S : sparse matrix (**dynamic information** on top of background)



Otazo, Candes, Sodickson 2014

Recent developments in CS MRI

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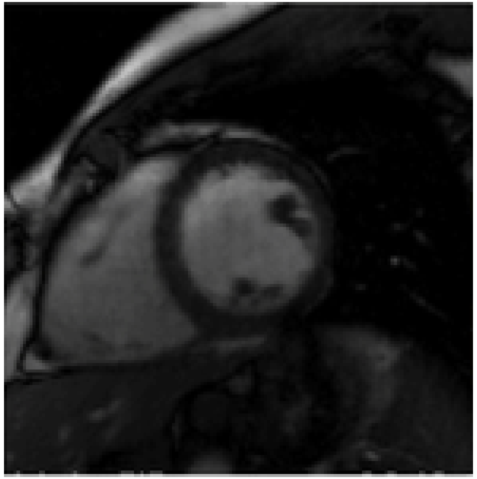


8-fold undersampling

Otazo, Candes, Sodickson 2014

Can we scan faster?

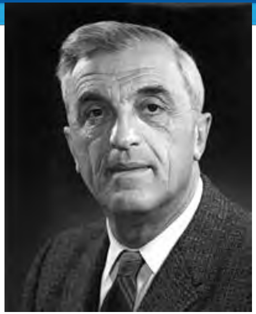
- So far, MRI reconstruction was more or less:

$$\min_{\mathbf{m}} \| F\mathbf{m} - \mathbf{d} \| + \lambda \| C\mathbf{m} \|$$

- Further undersampling by introducing physical constraints

$$\min_{\mathbf{m},\mathbf{p}} \| F\mathbf{m} - \mathbf{d} \| + \lambda \| C\mathbf{m} \|$$
$$\text{s.t. } g(\mathbf{m},\mathbf{p}) = 0 \text{ (physical model)}$$

The Bloch equation (1946)



Felix Bloch (1905-1983)
Nobel Prize in 1952

Local behaviour of nuclear magnetization:

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{m} = \begin{pmatrix} -\frac{1}{T_2} & \omega & \gamma b(t) \\ -\omega & -\frac{1}{T_2} & 0 \\ -\gamma b(t) & 0 & -\frac{1}{T_1} \end{pmatrix} \vec{m} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{T_1} \end{pmatrix}$$

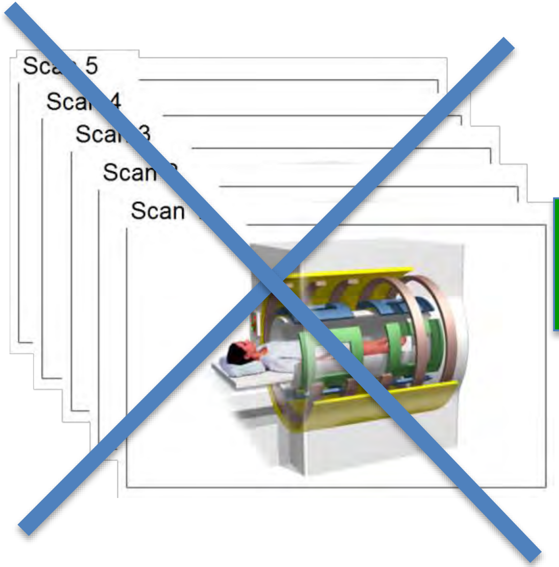
Experimental setting (t)

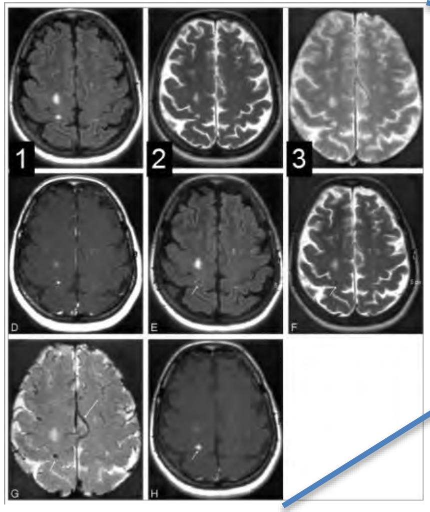
State (t,**r**)

Parameter (**r**)

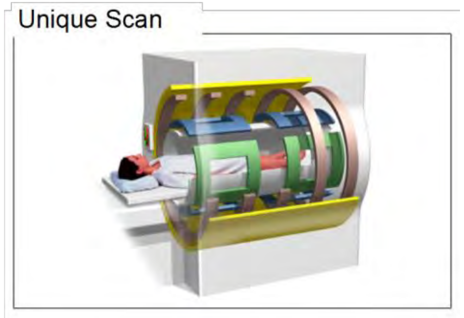
Unknown

State, m





FFT



Model Inversion

Forward model

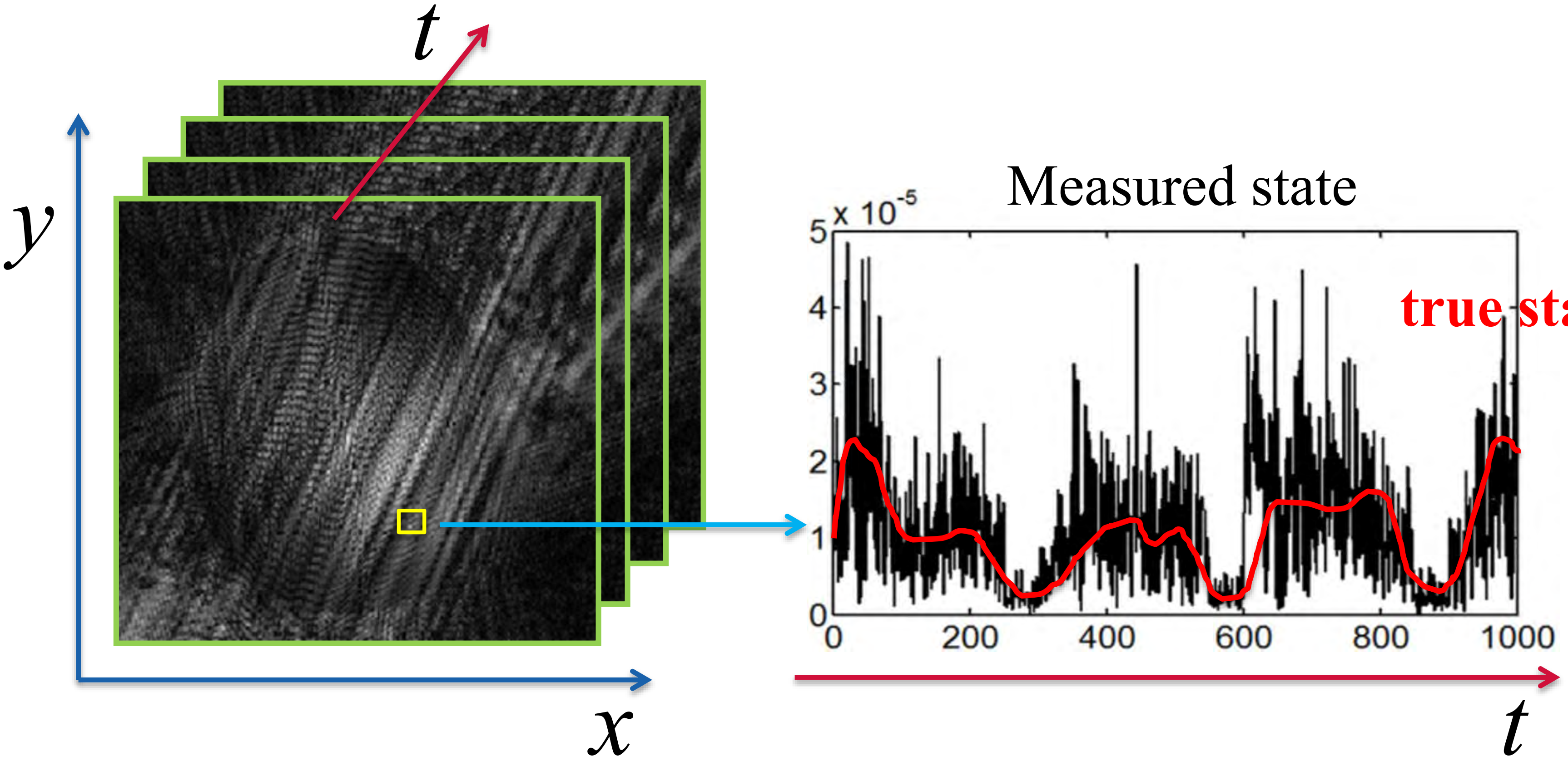
Quantitative Parameters

$$\min_{\mathbf{m},\mathbf{p}} \| F\mathbf{m} - \mathbf{d} \| + \lambda \| C\mathbf{m} \|$$
$$\text{s.t. } g(\mathbf{m},\mathbf{p}) = 0 \text{ (Bloch eq.)}$$

- So far, **linear** operators (FFT, Parallel imaging spatial weight functions, sparsify transforms …)

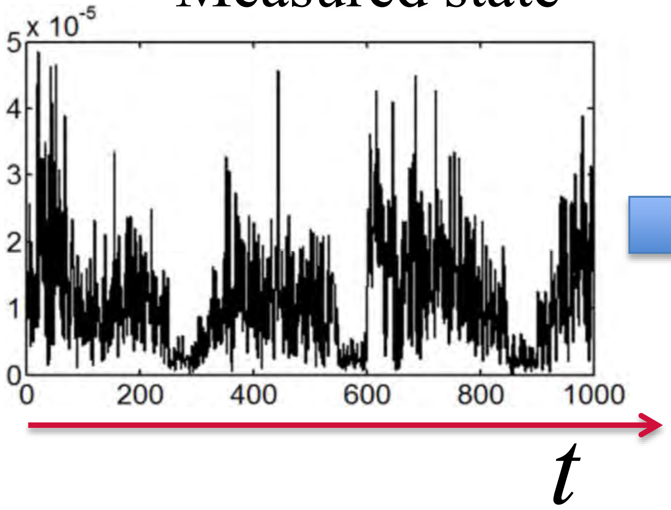
- Bloch equation is **nonlinear** in the model parameters
  1) computational complexity
  2) non-convexity

MR fingerprinting (Dan Ma *et al*, Nature 2013)



MR fingerprinting (Dan Ma *et al*, Nature 2013)

Measured state



Exhaustive search pre-computed dictionary of signal responses

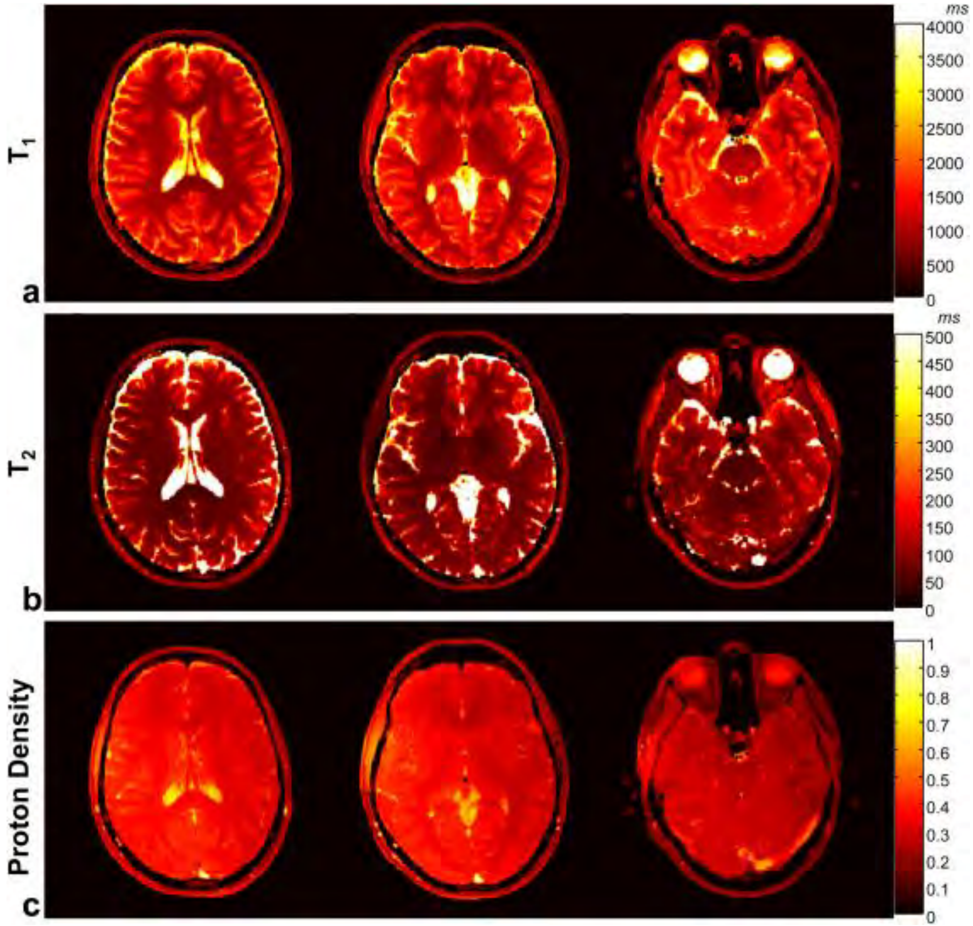
Parameters

T_1,

T_2,

ρ,

8 seconds scan



Jiang Y. *et al*, MRM 2015

MRF Dictionary. Computational complexity

Cost for constructing and storing the dictionary

# of parameters	Computing Time*	Memory*
4	17 hrs	400 Gb
6	43,000 hrs	960,000 Gb

Exponential growth for:

a) Computing time
b) Memory
c) Exhaustive search (matching)

Note:
SVD compression accelerates matching but growth is still **exponential**.
SVD of 1000 Tb matrix!

MRF without dictionary?

Measured time signal

Signal model (incl. slice profile)

$$\arg\min_{\rho,\beta} \sum_{j=1}^{N_t} |d(t_j) - \rho m(\beta, t_j)|^2$$

complex

T_1, T_2, ω, B_1

<u>Gradient-based iterations.</u> **Advantages:**

- **No dictionary**.
- Complexity is **linear** in # of parameters: Extra parameter → small increase

MRF without dictionary?

Measured time signal

Signal model (incl. slice profile)

$$\arg\min_{\rho,\beta} \sum_{j=1}^{N_t} |d(t_j) - \rho m(\beta, t_j)|^2$$

complex

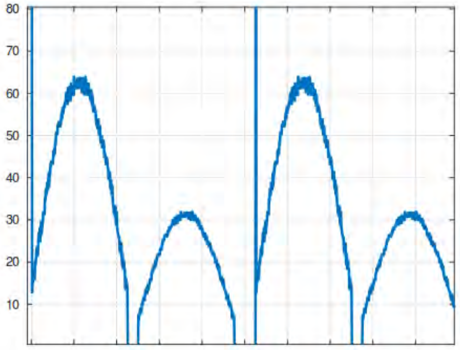
T_1, T_2, ω, B_1

<u>Gradient-based iterations.</u> **Tackle:**

1. **Poor performance** for large data-model discrepancy
2. **Local minima** for nonconvex functions.

Minimization landscape

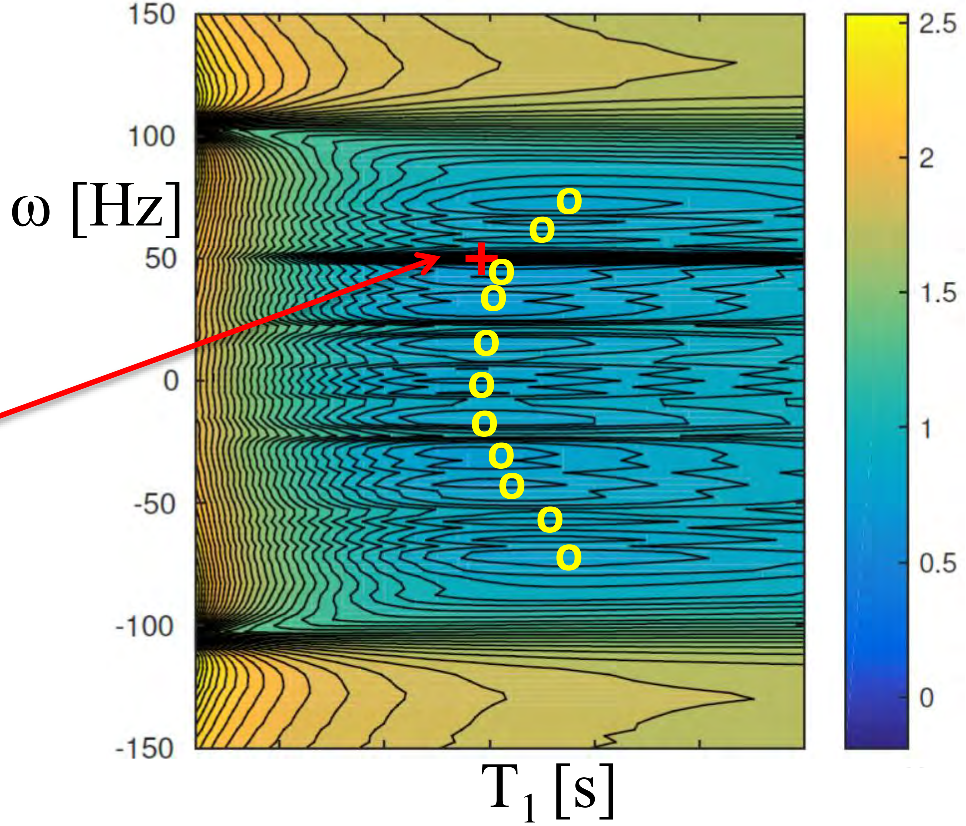
Experimental setting $\theta(t)$



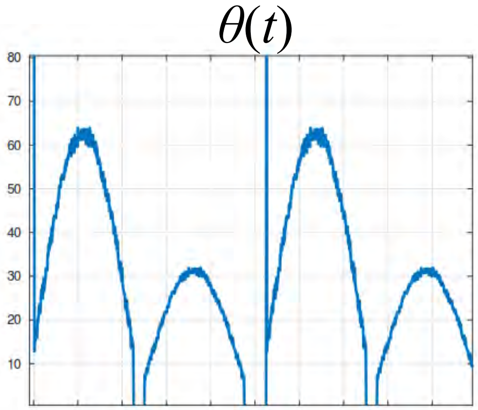
global minimum

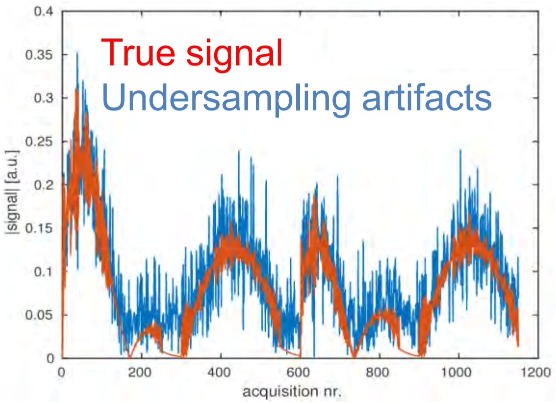
Many local minima!

Optimization landscape [a.u.]



Large data/model discrepancy in MRF





Bloch equation response and **undersampling artifacts** both **high frequency**:

→ difficult to **separate**, everything looks like noise

Low-pass experiment design

Find the experimental setting $\theta(t)$:

$$\arg\min_{\theta} \max_{n} \left\{ \frac{\sigma(T_1^n)}{T_1^n}, \frac{\sigma(T_2^n)}{T_2^n}, \frac{\sigma(\rho^n)}{\rho^n} \right\}$$

Sensitivity

$$\text{such that:} \quad \theta(0) = 0$$

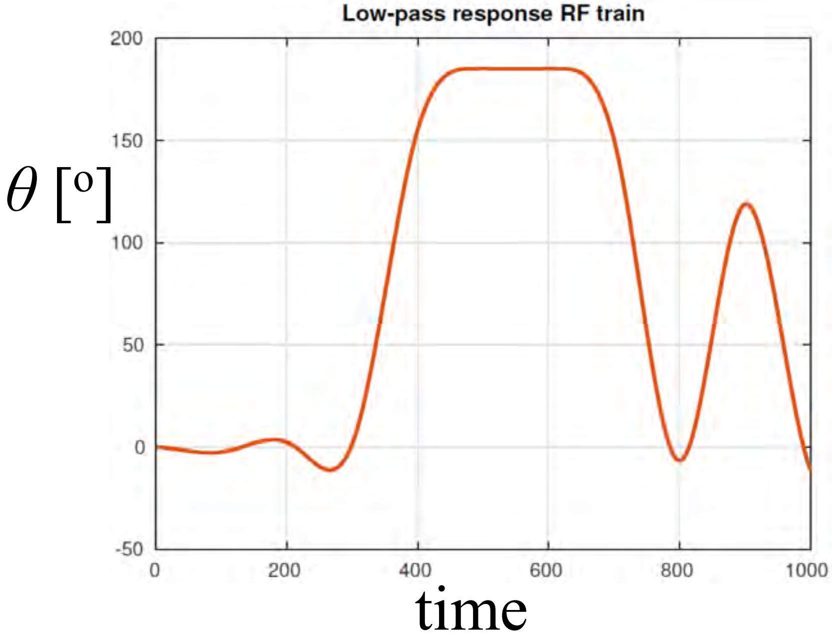
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \leq \delta$$

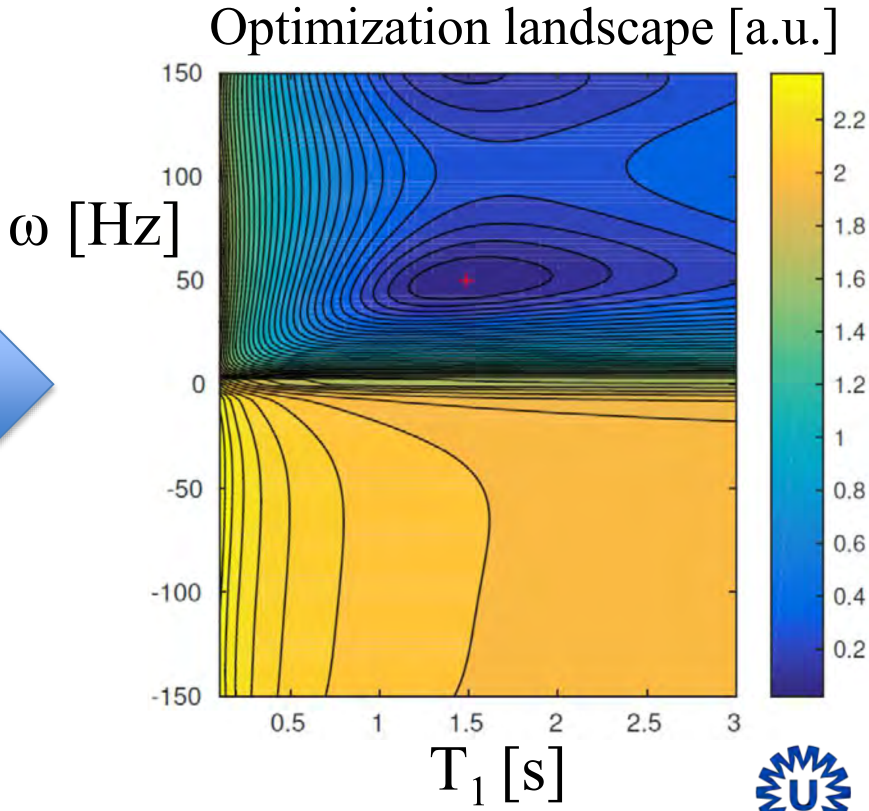
Smooth response

$$|\theta(t)| \leq \theta_{\max}$$

Physical limit

Sbrizzi, Bruijnen, van den Berg ISMRM 2017





Sbrizzi, Bruijnen, van den Berg ISMRM 2017

Model-based reconstruction

$$\min_{\mathbf{m},\mathbf{p}} \| F\mathbf{m} - \mathbf{d} \| + \lambda \| C\mathbf{m} \|$$
$$\text{s.t. } g(\mathbf{m},\mathbf{p}) = 0 \text{ (physical model)}$$

MR Fingerprinting approach:

$$\mathbf{m}^* = \arg\min_{\mathbf{m},\mathbf{p}} \| F\mathbf{m} - \mathbf{d} \| + \lambda \| C\mathbf{m} \|$$
$$\mathbf{p}_j^* = \arg\min \| g(\mathbf{m}_j^*, \mathbf{p}_j) \| \text{ for all } j$$

What about F ?

"it should not be taken too seriously" (Ljunggren, 1983)

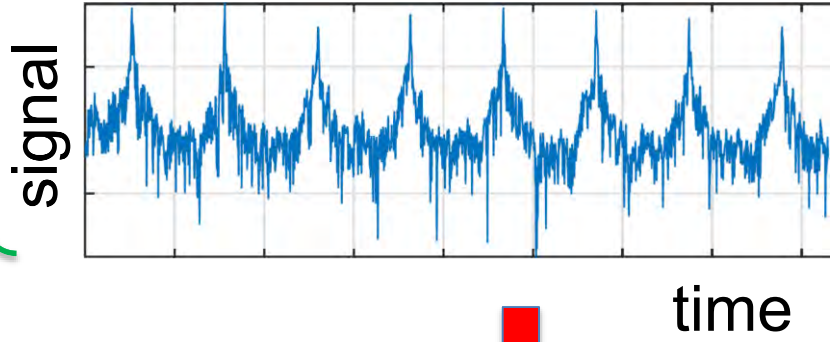
MR Spin TomogrAphy in Time-domain (MR-STAT)

Volumetric integral

Space/time dependence

Acquire signal:
Forward Model

$$s(t) = \int_V m_\perp(\vec{r}, t)\mathrm{d}\vec{r}$$



$T_1(\mathbf{r}), T_2(\mathbf{r}), \rho(\mathbf{r}),$
$\Delta B_0(\mathbf{r})\ldots$

Reconstruction:
Invert Model

Sbrizzi, van der Heide, van den Berg. arXiv 2017

MR-STAT reconstruction

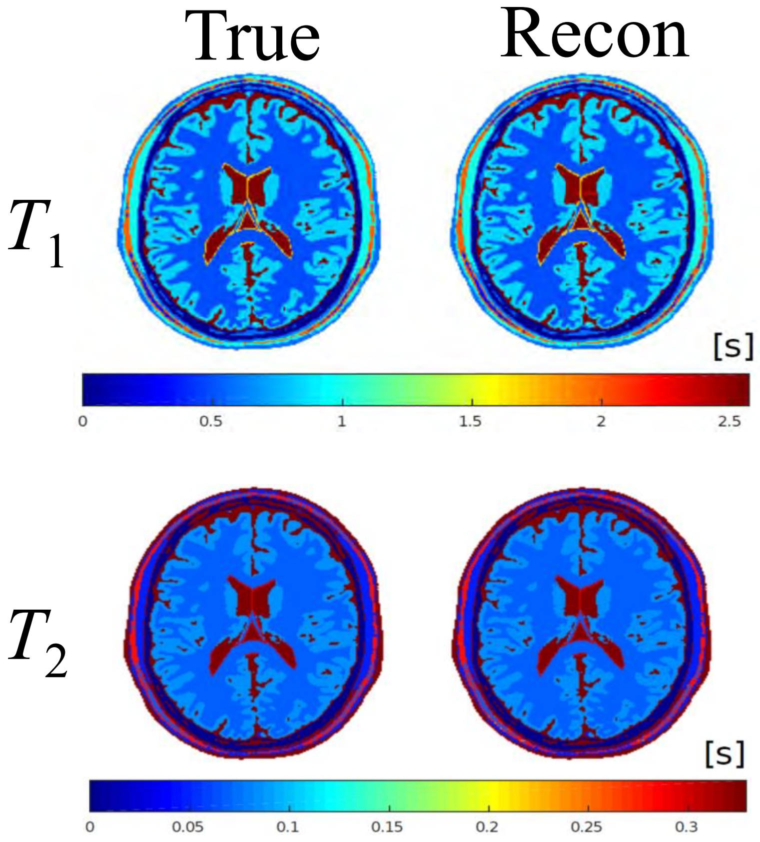
$$a(\vec{r}) \equiv |B_1^-(\vec{r})| M_0(\vec{r}) e^{\imath\phi(\vec{r})} \qquad \vec{b} = (T_1(\vec{r}), T_2(\vec{r}), |B_1^+(\vec{r})|, \Delta B_0(\vec{r}))$$

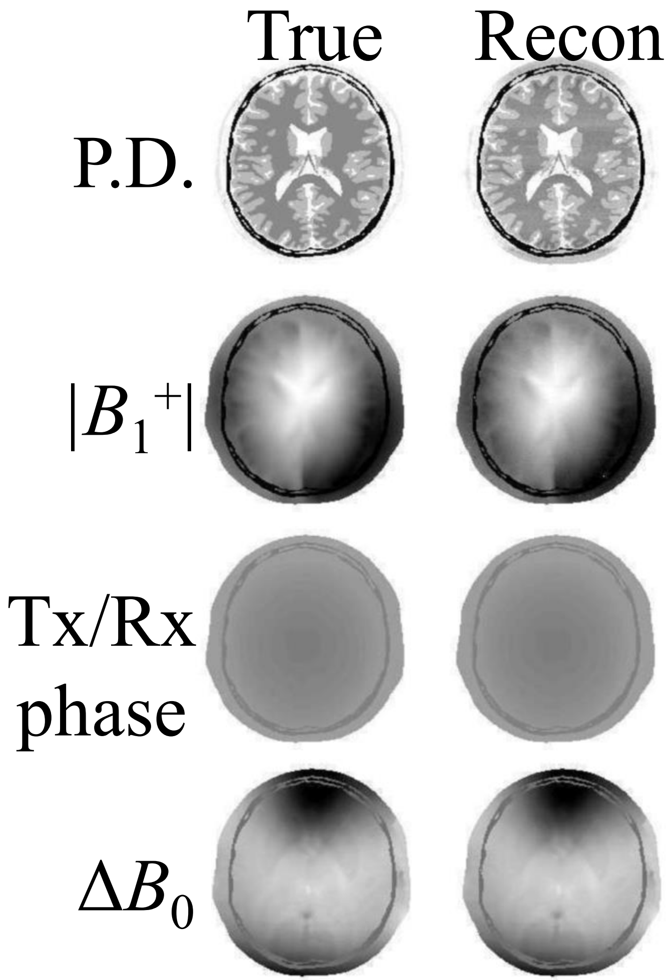
Time-data signal model

$$\begin{aligned}
(a^*, \vec{b}^*) \quad &= \arg\min_{a,\vec{b}} \quad \int_{t\in\tau} \left| d(t) - s(a,\vec{b},t) \right|^2 \mathrm{d}t, && \text{(Data consistency)}\\
\text{such that} \quad & s(a,\vec{b},t) = \int_V a\, m(\vec{b},t) \mathrm{d}\vec{r}, \quad t \in \tau && \text{(Faraday's law)}\\
& \tfrac{\mathrm{d}}{\mathrm{d}t}\vec{m} = \Pi\vec{m} + \vec{c} && \text{(Bloch equation)}\\
& \vec{m}(\vec{b},0) = \vec{e}_3 && \text{(Initial condition)}\\
& \vec{b} \in \mathbb{B} && \text{(Physical bounds)}
\end{aligned}$$

Sbrizzi, van der Heide, van den Berg. arXiv 2017

MR-STAT reconstruction





Sbrizzi, van der Heide, van den Berg. arXiv 2017

Summary

- ’70s and ’80s: toward a reliable, robust model. From radial projections to FFT. Experimental design is devoted to fit in the FFT framework (*k*-space)
- ’90s and ’00s: model is extended to parallel imaging and CS. FFT is still fundamental part of solution to linear systems. Iterative methods for large scale (ℓ_2/ℓ_1) regularized least squares.
- Present (and future?): physical modelling and simulations directly included in reconstruction. Nonlinearity, non-convexity, computational complexity. Numerical experimental design. AI.

Acknowledgments

NWO
Netherlands Organisation
for Scientific Research

Anna
Kruseman

Tom
Bruijnen

Oscar
van der Heide

MRI
scanner

Nico
van den Berg