Nonlinear Methods in Computational Finance – American Option Pricing

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15/10/2020

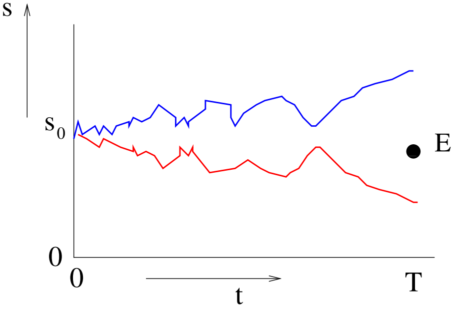
Agenda

- Option pricing, the mathematical framework
- Modeling American option with early-exercise feature
- Numerical techniques for such options
- Joint work with many PhD and MSc students!

Put option example

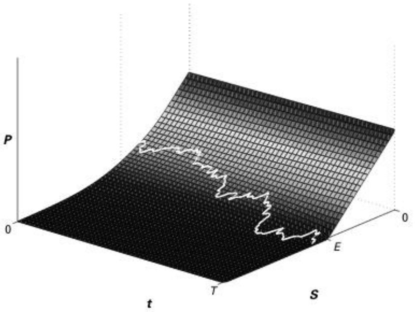
- Suppose you own stocks, and you'd like to have cash in two years.
- You need at least E euros for your stocks, but stocks may drop in the coming years. How to make sure that you'll receive at least E euros?
- Buy a put option (i.e. the right to sell stocks at a future time point for a fixed price E).

A put option gives the holder the right to sell in the future at a previously agreed strike price, E.



$v(T,S) = \max(E - S_T, 0) =: h(S_T)$

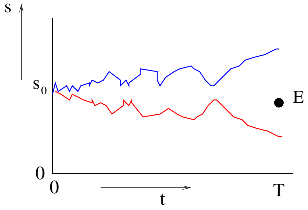
European put option



$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = 0$$

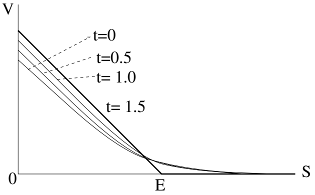
- Compute the put option surface (v for all S- and all t-values).
- Stock values S are modeled by a stochastic process.
- At any time, for any stock value, we'd know $v(t, S)$.

American options



- American options are contracts that may be exercised early, prior to expiry. Most traded stock options are American style.
- American option cannot be less than the equivalent European option.
- An important question: when should the option be exercised ?

American options



- European put is in a certain S range less than pay-off value, $v^{EU}(S,t) < \max(E-S,0)$.
- An American option can be exercised at any time. Buying the option v and the stock S, exercise the put immediately, i.e., sell the stock for E, gives a risk-free profit $E-v-S$, in this S-region !

$\Rightarrow$ When early exercise is permitted, a constraint must be imposed,

$$v^{AM}(S,t) \geq \max(E-S,0).$$

Basic free boundary example

- A contract without expiry time, $v(S)$ (solution independent of t).
- The option value cannot go below the payoff: $v \geq \max(E - S, 0)$.
- Since the option is independent of t, it must satisfy

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 v}{dS^2} + rS\frac{dv}{dS} - rv = 0.$$

  Solution of this ODE: $v(S) = AS + BS^{-2r/\sigma^2}$, with A, B constant.
- For the American put: $A = 0$, as $S \to \infty$ the value of the option must tend to zero.
- While the stock is 'high' don't exercise. If it falls low immediately exercise, receiving $E - S$.
- Let $S = S^*$ be the value at which we exercise, i.e. as soon as S reaches this value from above, we exercise.

Basic free boundary example

- At $S = S^*$ the option must equal the payoff, $v(S^*) = E - S^*$. It cannot be less (arbitrage), and if it is more, we would not exercise.
- The continuity of the option value with the payoff gives us,

$$v(S^*) = B(S^*)^{-2r/\sigma^2} = E - S^*.$$

- Eliminating B gives for $S > S^*$:

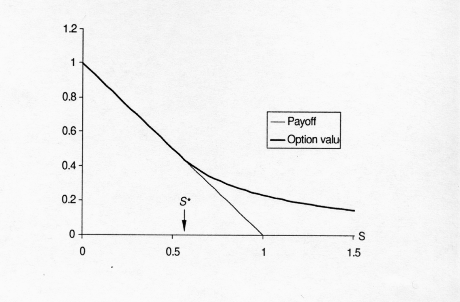
$$v(S) = (E - S^*)\left(\frac{S}{S^*}\right)^{-2r/\sigma^2}.$$

- Choose S^* to maximize the option value at any time before exercise. Differentiation with respect to S^*:

$$\frac{\partial}{\partial S^*}(E - S^*)\left(\frac{S}{S^*}\right)^{\frac{-2r}{\sigma^2}} = \frac{1}{S^*}\left(\frac{S}{S^*}\right)^{\frac{-2r}{\sigma^2}}\left(-S^* + \frac{2r}{\sigma^2}(E - S^*)\right) = 0$$

Basic example

- One finds, $S^* = \frac{E}{1+\sigma^2/2r}$, which maximizes $v(S)$ for all $S \geq S^*$.



- The slope of the option value and of the payoff function are the same.
- The American option is maximized by an exercise strategy that makes the option value and the slope continuous.
- Position S^* is called the optimal exercise point.

Writer/Holder

- The writer assumes that the holder will exercise at the worst possible time for the writer.
- She assumes that the option is exercised at the moment that gives the writer the least profit.

⇒ This is often referred to as "the optimal stopping time"

- Out of all strategies one must find the one that gives the option the least value to the writer.

- The contract and option value will be functions of S and t, $v = v(t, S)$.
- In the region where $v = E - S$, $(S < E)$, we find by substitution:

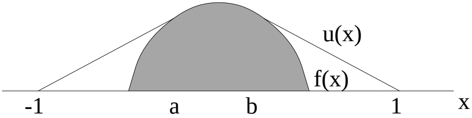
$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv = -rE < 0$$

- Inequality can also be derived with standard Black-Scholes analysis with minor modifications.
- This gives us the following inequality for the whole S-region:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv \leq 0$$

- The problem to solve for an American put option contract:

$$
\begin{aligned}
\frac{\partial v}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv \leq 0 \\
v(t,S) &\geq \max(E-S,0) \\
v(T,S) &= \max(E-S,0) \\
\frac{\partial v}{\partial S} & \quad \text{is continuous}
\end{aligned}
$$



- Given: an "obstacle" $h(x)$, with $h(x) > 0$ for $a < x < b, h \in C^2,\ h'' < 0$ and $h(-1) < 0, h(1) < 0$.
- One spans a function u of minimal length; on the obstacle between a and b. The obstacle problem is a free boundary problem:

$$\begin{aligned} -1 < x < a: &\quad u'' = 0 \quad (u > h) \\ a < x < b: &\quad u = h \quad (u'' = h'' < 0) \\ b < x < 1: &\quad u'' = 0 \quad (u > h) \end{aligned}$$

Linear complementarity

- $u > h$, then $u'' = 0$
  $u = h$, then $u'' < 0$.
- Reformulation of the obstacle problem:
  Find $u(x)$, so that

$$\begin{aligned} u''(u-h) &= 0, \qquad -u'' \geq 0, \quad u - h \geq 0, \\ u(-1) = u(1) &= 0, \qquad u \in C^1[-1,1] \end{aligned}$$

- A similar complementarity follows for the American options:
  $v > \max(E - S, 0), \quad S > S^*(t)$, then Black-Scholes equation,
  $v = E - S, \quad S \leq S^*(t)$, then Black-Scholes inequality

LCP for American option

- When the asset S follows GBM: $dS = \mu S dt + \sigma S dW$, the American pricing problem can be formally stated as an LCP:

$$\begin{aligned} Lv &\geq 0 \\ (v-h) &\geq 0 \\ (Lv = 0) &\vee (v - h = 0) \end{aligned}$$

- The notation $(Lv = 0) \vee (v - h = 0)$ denotes that either $(Lv = 0)$ or $(v - h = 0)$ at each point in the solution domain, and

$$Lv \equiv \frac{\partial v}{\partial \tau} - (\frac{\sigma^2}{2} S^2 \frac{\partial^2 v}{\partial S^2} + rS \frac{\partial v}{\partial S} - rv)$$

Penalty formulation

- The basic idea of the penalty method is simple: We replace the LCP problem by a nonlinear PDE:

$$\frac{\partial v}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 v}{\partial S^2} + rS \frac{\partial v}{\partial S} - rv + \frac{1}{\epsilon} \max(h - v, 0),$$

  where the positive penalty parameter $\epsilon \to 0$ effectively ensures that the solution satisfies $v \geq h$.
- We can use iterative methods for the numerical solution (GMRES, etc.)

American option, dynamic programming

- The value function v is related to the HJB variational inequality:

$$\min[-\frac{\partial v}{\partial t} - \mathcal{L}v, v - h] = 0,$$

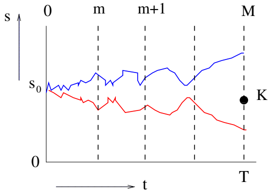
- C is the continuation region, the complement set is the stopping or exercise region (receive the reward h).
- With $t \in [0, T]$, and stopping times $\mathcal{T}_{t,T}$, the finite horizon optimal stopping problem is formulated as

$$v(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r(\tau - t)} h(X_\tau)\right].,$$

where the controller only has control over her terminal time, and

$$dX_t = \mu(X_t)dt + \sigma(X_t)d\omega_t,$$

Pricing options with early-exercise



- The pricing formulas for a Bermudan option with M exercise dates reads, for $m = M-1, \ldots, 1$:

$$\begin{cases} c(t_m, S) & = & e^{-r\Delta t_m}\mathbb{E}\left[v(t_{m+1}, S)|S_{t_m}\right], \\ v(t_m, S) & = & \max\left(h(S_{t_m}), c(t_m, S)\right) \end{cases}$$

and $v(t_0, S) = e^{-r\Delta t_0}\mathbb{E}\left[v(t_1, S)|S_{t_0}\right]$

Conclusion

- American option, with early-exercise feature, is the fundamental nonlinear problem in finance
- Free boundary problem, reformulated as an LCP, or penalty formulation
- We use backward dynamic programming, with conditional expectations to solve such problems
- Can be generalized to high-D problems.

Contributions Computational Finance (2000-...)