Market risk measures with stochastic liquidity horizon by Shannon wavelet expansions

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Motivation
Risk measures

**Definition**

Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a time horizon \(\Delta t\). Denote by \(\mathcal{L}\) the set of all random variables on \((\Omega, \mathcal{F})\) (representing the portfolio returns/loses over a time horizon \(\Delta t\)). Then, **risk measures** are real-valued maps \(\rho : \mathcal{L} \rightarrow \mathbb{R}\).

A risk measure is **coherent** if it satisfies: normality, monotonicity, sub-additivity, positive homogeneity and translation invariance.

- **Use**: determine the amount of currency to keep in reserve.
- **Purpose**: make the risks taken by financial institutions \(\{\text{banks, insurance companies}\}\) acceptable to the regulator.
- The most famous: **VaR** and **ES**.
Value at Risk (VaR) and Expected Shortfall (ES)

**Definition**

Given a confidence level $\alpha \in (0, 1)$. Being $L$ a loss.

The $\text{VaR}_\alpha$ is the smallest number $l$ such that the probability that the loss $L$ exceeds $l$ is at most $(1 - \alpha)$. I.e.

$$\text{VaR}_\alpha(L) = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\}.$$ 

The $\text{ES}_\alpha$ is defined by

$$\text{ES}_\alpha(L) := \frac{1}{1 - \alpha} \int_\alpha^1 \text{VaR}_u(L) \, du.$$ 

- VaR is a quantile of the loss distribution.
- VaR is not a coherent risk measure, not satisfies sub-additivity.
- ES is more sensitive to the shape of the loss distribution in the tail of the distribution. ES is a coherent risk measure.
The financial crisis, review of BCBS

Basel Committee of Banking Supervision (BCBS) stated:

- The crisis exposed:
  - Weaknesses in the framework design for capitalizing trading activities.
  - Insufficient capital level required against trading book exposures to absorb losses.

- Review/assessment:
  - From VaR to ES, due to the inability to capture the risk in the tail.
  - Incorporate market liquidity risk. The time it takes to liquidate a risk position varies; thus, the horizon should be extended.
Our purpose

- Produce a **set of numerical techniques** to address the challenge of the VaR and ES computation under a stochastic liquidity horizon framework (idea from Brigo and Nordio, 2015).

- To do so, we use **SWIFT**. Because:
  - In the scenarios we work, the **characteristic function** of the density is known. Thus, makes sense a Fourier inversion method.
  - Densities with stochastic holding period have fat tails, so we do no need to rely on a **truncation range**.
  - Make use of **wavelets** properties to get the risk measures values.
  - Analysis of the **error** is available.
  - There are rules on how to select the **parameters**.
SWIFT
Shannon Wavelet Inverse Fourier Technique
Definition

Let $V_j, j = \cdots, -2, -1, 0, 1, 2, \cdots$ be a sequence of subspaces of functions in $L^2(\mathbb{R})$. The collection of spaces $(V_j)_{j \in \mathbb{Z}}$ is called a multiresolution analysis (MRA) of $L^2(\mathbb{R})$ with scaling function $\phi \in V_0$, if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$,
2. (dense) $\overline{\bigcup V_j} = L^2(\mathbb{R})$,
3. (separation) $\cap V_j = \{0\}$,
4. (scaling) The function $f(x)$ belongs to $V_j$ if and only if the function $f(2x)$ belongs to $V_{j+1}$,
5. (orthonormal basis) The function $\phi$ belongs to $V_0$ and the set $\{\phi(x - k), k \in \mathbb{Z}\}$ is an orthonormal basis (using the $L^2$ inner product) for $V_0$.

MRA defines general wavelet structures in $L^2(\mathbb{R})$. 
The set of functions

\[ \{ \phi_{m,k}(x) = 2^{m/2} \phi(2^m x - k); k \in \mathbb{Z} \} \]

is an orthonormal basis for \( V_m \).

**Lemma**

Let us define \( P_m f \) as the orthogonal projection of a function \( f \in L^2(\mathbb{R}) \) on the space \( V_m \), constructed by

\[ P_m f(x) = \sum_{k \in \mathbb{Z}} c_{m,k} \phi_{m,k}(x), \]

where \( c_{m,k} = \int_{\mathbb{R}} f(x) \bar{\phi}(x) dx \). Then, the convergence of the projection \( P_m f(x) \) holds in the \( L^2(\mathbb{R}) \) – norm.
Shannon wavelets

- **Cardinal sine function (sinc):**

\[ \phi(x) = \text{sinc}(x) := \frac{\sin(\pi x)}{\pi x} \]  
(Shannon scaling function).

- **Simplicity in the Fourier domain:**

\[ \hat{\phi}(\omega) := \int_{\mathbb{R}} \phi(x)e^{-i\omega x} \, dx = \text{rect} \left( \frac{\omega}{2\pi} \right). \]
Let us consider a density function \( f \in L^2(\mathbb{R}) \). Assuming \( \hat{f} \) to be known. Following MRA we approximate \( f \) by \( f_m \):

\[
f(x) \approx f_m(x) := \sum_{k=k_1}^{k_2} c_{m,k}^* \phi_{m,k}(x),
\]

where \( c_{m,k}^* \approx c_{m,k} = \langle f, \phi_{m,k} \rangle \) (scaling coefficients).

Approximation technique:

- **Step 1:** Projection on the space \( V_m \) (seen).
- **Step 2:** Truncation of the infinite sum.
- **Step 3:** Approximation of the scaling coefficients by assuming known the characteristic function of \( f \).
Lemma

The scaling coefficients \( c_{m,k} \) satisfy,

\[
\lim_{k \to \pm \infty} c_{m,k} = 0.
\]
Proof.

The set of Shannon scaling functions in $V_m$ is defined as

$$\phi_{m,k}(x) = 2^{m/2} \frac{\sin(\pi(2^m x - k))}{\pi(2^m x - k)}, \quad k \in \mathbb{Z}.$$  

Thus, for $h \in \mathbb{Z}$,

$$\phi_{m,k} \left( \frac{h}{2^m} \right) = 2^{m/2} \delta_{k,h},$$

being $\delta_{k,h}$ the Kronecker delta. It gives us that

$$P_m f \left( \frac{h}{2^m} \right) = 2^{m/2} \sum_{k \in \mathbb{Z}} c_{m,k} \delta_{k,h} = 2^{m/2} c_{m,h}.$$  

Since $f$ is a density function, we assume it to tend to zero at plus and minus infinity.
To do so, we make use of Vieta’s formula.

Vieta’s formula

\[
sinc(t) := \frac{\sin(\pi t)}{(\pi t)} \approx \frac{1}{2^{J-1}} \sum_{j=1}^{2^{J-1}} \cos \left( \frac{2j - 1}{2^J} \pi t \right).
\]

Using Vieta’s formula and some algebraic manipulation, one arrives to the coefficients expression.

Coefficients approximation

\[
c_{m,k} \approx c_{m,k}^* := \frac{2^{m/2}}{2^{J-1}} \sum_{j=1}^{2^J} \text{Re} \left[ \hat{f} \left( \frac{(2j - 1) \pi 2^m}{2^J} \right) e^{\frac{ik\pi (2j-1)}{2^J}} \right].
\]

- Note the need of the characteristic function.
We can evaluate the density at the extremes of the interval and compute the area underneath the density function as a byproduct, since

\[ f_m \left( \frac{h}{2^m} \right) = 2^{\frac{m}{2}} c_{m,k}, \ h \in \mathbb{Z}. \]

Then

\[ A = \frac{1}{2^{\frac{m}{2}}} \left( \frac{c_{m,k_1}}{2} + \sum_{k_1 < k < k_2} c_{m,k} + \frac{c_{m,k_2}}{2} \right). \]
Efficient computation of liquidity-adjusted risk measures
We recover the density function of the portfolio change $\Delta V$ from its Fourier transform, carrying out the Fourier inversion by means of SWIFT.

We speed up the computation by using a FFT algorithm.

We look for the $\alpha$-quantile of the distribution. To do so:

1. We find $h$ and $h+1$ such that $2^m$ VaR is located between these two values (it is a sum of trapezoids).

2. We can accurately compute the VaR using a bisection method within the interval $\left[ \frac{h}{2^m}, \frac{h+1}{2^m} \right]$. 
Using Vieta’s formula

\[
\text{ES}(\alpha) = \frac{1}{1 - \alpha} \int_{\text{VaR}(\alpha)}^{+\infty} xf(x) \, dx
\]

\[
\approx \frac{1}{1 - \alpha} \int_{\text{VaR}(\alpha)}^{b} x \sum_{k=k_1}^{k_2} c_{m,k} \phi_{m,k}(x) \, dx.
\]
Let us assume we have the Fourier transform of the deterministic situation: \( \hat{f}_{\Delta V} \).

We assume that the stochastic holding period \( \Delta t \) follows a process with density function \( f_{\Delta t} \).

Making use of the rule \( \mathbb{E} [ \mathbb{E} [X \mid Y]] = \mathbb{E} [X] \), we have

\[
\hat{f}_{\Delta V(\Delta t)} (u) = \int_{\mathbb{R}} \hat{f}_{\Delta V} (u) f_{\Delta t} (h) \, dh.
\]

Then, using a numerical integration quadrature we compute VaR and ES as in the deterministic situation.
There exists closed form solution.

The characteristic function of the log-return portfolio change is

\[ \hat{f}_{\Delta X_{\Delta t}}(u) = e^{-i\mu u \Delta t - \frac{(\sigma u)^2}{2} \Delta t}. \]

(a) $\Delta t = 1/365$.

(b) $\Delta t \sim \exp(10)$. 
Results: Under delta-gamma approach (1)

Delta-gamma approximation

It consists of approximate the change in a portfolio value $\Delta V$ by

$$
\Delta V \approx \Delta V_\gamma := \Theta \Delta t + \delta^T \Delta S + \frac{1}{2} \Delta S^T \Gamma \Delta S,
$$

where $S(t) = (S_1(t), \cdots, S_p(t))^T$ are the risk factors, $\Theta = \frac{\partial V}{\partial t}$, $\delta_i = \frac{\partial V}{\partial S_i}$ and $\Gamma_{i,j} = \frac{\partial^2 V}{\partial S_i \partial S_j}$.

(Mathai and Provost, 1992) It is known the characteristic function of $f_{\Delta V_\gamma}$ under the assumption that $\Delta S$ follows a normal distribution.
Table: Bernoulli SLH. Reference prices by Monte Carlo.

<table>
<thead>
<tr>
<th>Holding Period</th>
<th>Prob - case 1</th>
<th>Prob - case 2</th>
<th>Prob - case 3</th>
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<tbody>
<tr>
<td>10</td>
<td>0.25</td>
<td>0.5</td>
<td>0.75</td>
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<tr>
<td>30</td>
<td>0.75</td>
<td>0.5</td>
<td>0.25</td>
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<tr>
<td>VaR</td>
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<td>3.0418</td>
<td>3.0364</td>
</tr>
<tr>
<td>ES</td>
<td>3.0436</td>
<td>3.0432</td>
<td>3.0414</td>
</tr>
</tbody>
</table>

Risk measures log10-errors

![Graph showing VaR and ES errors for different holding periods and probabilities](image-url)
Conclusions
Conclusions

- SWIFT method has been presented.
- SWIFT method has been used to compute VaR and ES.
- The holding period in VaR and ES has been considered stochastic to reflect the liquidity risk.
- We exhibited the convergence of the method by means of some examples.
Thank you! 😊😊😊