The COS method for option valuation under the SABR dynamics

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May 23, 2016

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SABR model

Introduced in 2002 by Hagan, Kumar, Lesniewski and Woodward [1].
Volatility α_t is a random function over time and F_t denotes the forward process:

$$
\begin{aligned}
\mathrm{d}F_t &= \alpha_t (F_t)^{\beta} \, \mathrm{d}W_t^1, & F_0 &= f, \\
\mathrm{d}\alpha_t &= \rho\nu\alpha_t \, \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\nu\alpha_t \, \mathrm{d}W_t^2, & \alpha_0 &= \alpha.
\end{aligned}
$$

Constraints

- $f \geq 0$
- $\nu \geq 0$
- $\alpha \geq 0$
- $0 \leq \beta \leq 1$
- $-1 \leq \rho \leq 1$

[1] P.S. Hagan, D. Kumar, A.S. Lesniewski and D.E. Woodward. Managing Smile Risk. *Wilmott Magazine*, 84-108, September 2002.

Risk neutral option price

A European call option gives the owner the right to buy a stock at time T for price K.

Price $V(t_0 = 0, K)$ of this option under the risk-neutral measure:

$$V(0, K) = \exp(-rT)\mathbb{E}\left[\max\{F_T - K, 0\} \mid F_0 = f, \alpha_0 = \alpha\right],$$

where r is the continuous risk-free interest rate.

The discounted option price is a martingale.

Characteristic function

Option price

$$\begin{aligned} V(0,K) &= \exp(-rT)\mathbb{E}\left[\max\left\{F_T - K, 0\right\} \middle| f, \alpha\right] \\ &= \exp(-rT)\int_{\mathbb{R}^2} \max\left\{F - K, 0\right\} q_{F_T,\alpha_T}(F, A|f,\alpha)\,\mathrm{d}F\,\mathrm{d}A, \end{aligned}$$

where $q_{F_T,\alpha_T}(F, A|f,\alpha)$ is the conditional PDF of $F_T = F$ and $\alpha_T = A$ given $F_0 = f$ and $\alpha_0 = \alpha$.

The characteristic function (ChF) is the inverse Fourier transform of the PDF:

$$\phi_{F_T,A_T}(\mathbf{u}|f,\alpha) = \int_{\mathbb{R}^2} \exp\left(i\mathbf{u}^T\left[F \;\; A\right]\right) q_{F_T,A_T}(F, A|f,\alpha)\,\mathrm{d}F\,\mathrm{d}A$$

We truncate the infinite integration domain to a finite domain $[a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, such that

$$
\begin{aligned}
V(0,K) &= \exp(-rT) \int_{\mathbb{R}^2} \max\{F_T - K, 0\}\, q_{F_T,A_T}(F, A|f,\alpha)\, \mathrm{d}F\, \mathrm{d}A \\
&\approx \exp(-rT) \int_{a_2}^{b_2} \int_{a_1}^{b_1} \max\{F_T - K, 0\}\, q_{F_T,A_T}(F, A|f,\alpha)\, \mathrm{d}F\, \mathrm{d}A
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{F_T,A_T}(\mathbf{u}|f,\alpha) &= \int_{\mathbb{R}^2} \exp\left(i\mathbf{u}^T [F \;\; A]\right) q_{F_T,A_T}(F, A|f,\alpha)\, \mathrm{d}F\, \mathrm{d}A \\
&\approx \int_{a_2}^{b_2} \int_{a_1}^{b_1} \exp\left(i\mathbf{u}^T [F \;\; A]\right) q_{F_T,A_T}(F, A|f,\alpha)\, \mathrm{d}F\, \mathrm{d}A
\end{aligned}
$$

are well approximated.

We use Fourier-cosine expansion, and choose an $N \in \mathbb{N}$ such that

$$
\begin{aligned}
q_{F_T,A_T}(F, A|f, \alpha) &= \sum_{k_1=0}^{\infty}{}' \sum_{k_2=0}^{\infty}{}' \mathcal{Q}_{k_1,k_2} \cos\left(k_1\pi \frac{F-a_1}{b_1-a_1}\right) \cos\left(k_2\pi \frac{A-a_2}{b_2-a_2}\right) \\
&\approx \sum_{k_1=0}^{N}{}' \sum_{k_2=0}^{N}{}' \mathcal{Q}_{k_1,k_2} \cos\left(k_1\pi \frac{F-a_1}{b_1-a_1}\right) \cos\left(k_2\pi \frac{A-a_2}{b_2-a_2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{Q}_{k_1,k_2} &= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2} \int_{a_2}^{b_2}\int_{a_1}^{b_1} q_{F_T,A_T}(F, A|f, \alpha) \\
&\cdot \cos\left(k_1\pi \frac{F-a_1}{b_1-a_1}\right) \cos\left(k_2\pi \frac{A-a_2}{b_2-a_2}\right) \mathrm{d}F\,\mathrm{d}A.
\end{aligned}
$$

We use the relation

$$\cos(x) = \Re[\exp(ix)]$$

and

$$\mathcal{Q}_{k_1,k_2} = \frac{1}{2}\left[\mathcal{Q}^{+}_{k_1,k_2} + \mathcal{Q}^{-}_{k_1,k_2}\right],$$

where

$$\begin{aligned}
\mathcal{Q}^{\pm}_{k_1,k_2} &= \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} q_{F_T,A_T}(F, A | f, \alpha) \\
&\cdot \cos\left(k_1\pi\frac{F - a_1}{b_1 - a_1} \pm k_2\pi\frac{A - a_2}{b_2 - a_2}\right) \mathrm{d}F\,\mathrm{d}A \\
&\approx \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2}\mathcal{R}\left\{\phi_{F_T,A_T}\left(\frac{k_1\pi}{b_1 - a_1}, \pm\frac{k_2\pi}{b_2 - a_2}\middle| f, \alpha\right)\right. \\
&\cdot \left.\exp\left(-ik_1\pi\frac{a_1}{b_1 - a_1} \mp ik_2\pi\frac{a_2}{b_2 - a_2}\right)\right\}
\end{aligned}$$

Combining everything gives:

$$
\begin{aligned}
V(0,K) &\approx \exp(-rT)\int_{a_2}^{b_2}\int_{a_1}^{b_1}\max\{F_T-K,0\}\,q_{F_T,A_T}(F,A|f,\alpha)\,\mathrm{d}F\,\mathrm{d}A \\
&\approx \exp(-rT)\int_{a_2}^{b_2}\int_{a_1}^{b_1}\max\{F_T-K,0\}\sum_{k_1=0}^{N}{}'\sum_{k_2=0}^{N}{}'\mathcal{Q}_{k_1,k_2} \\
&\cdot \cos\left(k_1\pi\frac{F-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{A-a_2}{b_2-a_2}\right)\mathrm{d}F\,\mathrm{d}A \\
&\approx \exp(-rT)\sum_{k_1=0}^{N}{}'\sum_{k_2=0}^{N}{}'\mathcal{Q}_{k_1,k_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1}\max\{F_T-K,0\} \\
&\cdot \cos\left(k_1\pi\frac{F-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{A-a_2}{b_2-a_2}\right)\mathrm{d}F\,\mathrm{d}A
\end{aligned}
$$

COS method [2,3]

$$= \exp(-rT) \sum_{k_1=0}^{N}{}' \sum_{k_2=0}^{N}{}' \frac{1}{2} \left[\mathcal{Q}^{+}_{k_1,k_2} + \mathcal{Q}^{-}_{k_1,k_2} \right] \mathcal{V}_{k_1,k_2}$$

where

$$\begin{aligned} \mathcal{V}_{k_1,k_2} &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} \max\{F_T - K, 0\} \cos\left(k_1 \pi \frac{F - a_1}{b_1 - a_1}\right) \\ &\cdot \cos\left(k_2 \pi \frac{A - a_2}{b_2 - a_2}\right) \mathrm{d}F \, \mathrm{d}A \end{aligned}$$

[2] F. Fang and C.W. Oosterlee. A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions. *SIAM Journal on Scientific Computing*, **31**, pp. 826-848, 2008.

[3] M.J. Ruijter and C.W. Oosterlee. Two-dimensional Fourier Cosine Series Expansion Method for Pricing Financial Options. *SIAM Journal on Scientific Computing*, **34**(5):B642-B671, 2012.

COS method

Advantages:

- $\mathcal{V}_{k_1,k_2}$ can be evaluated efficiently.
- Often no analytical formula for the PDF available,
  but there is for the ChF.
- For small time steps the ChF is more stable (uncertainty principle).

COS method for SABR

The ChF of the SABR model is not available, so we can not use the general COS method for pricing derivatives.

We discretize the SABR FSDEs [4], with for example the Euler scheme [5]:

$$\begin{aligned} F^{\Delta}_{m+1} &= F^{\Delta}_{m} + \alpha^{\Delta}_{m}\left(F^{\Delta}_{m}\right)^{\beta}\Delta W^{1}_{m+1}, \\ \alpha^{\Delta}_{m+1} &= \alpha^{\Delta}_{m} + \rho\nu\alpha^{\Delta}_{m}\Delta W^{1}_{m+1} + \sqrt{1-\rho^2}\nu\alpha^{\Delta}_{m}\Delta W^{2}_{m+1}, \end{aligned}$$

for $m = 0, 1, ..., M-1$, $\Delta t = \frac{T}{M}$, and ΔW^{1}_{m+1} and ΔW^{2}_{m+1} are i.i.d. $\mathcal{N}(0, \Delta t)$. The bivariate ChF of the discretized FSDEs is analytically known. Also for other schemes, such as the 2.0-weak-Taylor scheme.

[4] M.J. Ruijter and C.W. Oosterlee. Numerical Fourier Method and second-order Taylor Scheme for Backward SDEs in Finance. *Applied Numerical Mathematics*, **103**:126, 2016.

[5] P.E. Kloeden and E. Platen. *Numerical Solution of Stocastic Differential Equations*. Springer, first edition, 1992.

COS method for SABR

We use the ChF of the discretized process and evaluate the COS method on a two-dimensional grid backwards in time.

The SABR has a unique property: we can price European options under this model for multiple strikes at once [6].

We use Richardson extrapolation on the Euler results to obtain second order convergence.

[6] H. Park. Efficient Valuation Method for the SABR Model. *SSRN paper*, November 2013.

Example: SABR

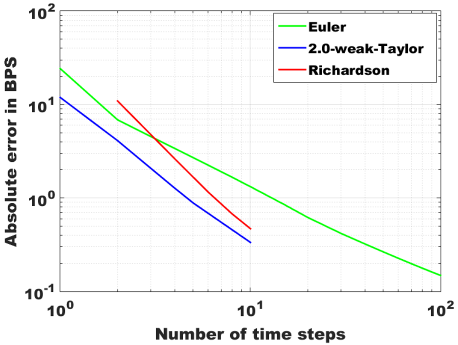

Figure: European call option with $f = 2$, $\alpha = 0.35$, $\beta = 0.8$, $\rho = 0$, $\nu = 0.4$, $T = 1$, strikes $K = 2.55, 2.7, 2.85$.

Example: SABR

	M	CPU	Absolute error in BPS
Euler	28	65.41s	< 1
2.0-weak-Taylor	5	143.56s	< 1
Richardson	7	19.48s	< 1
Euler	56	134.85s	< 0.5
2.0-weak-Taylor	7	235.42s	< 0.5
Richardson	9	26.86s	< 0.5

The 2.0-weak-Taylor scheme is slow, we use Richardson extrapolation on the Euler results to gain a significant reduction in CPU time **and** second order convergence.

Heston model

Reference model for convergence purposes: The Heston model.

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}A_t\,\mathrm{d}t + \sqrt{A_t}\,\mathrm{d}W_t^1, & X_0 = \log(s),\\
\mathrm{d}A_t &= \rho\nu\sqrt{A_t}\,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\nu\sqrt{A_t}\,\mathrm{d}W_t^2, & A_0 = a,
\end{aligned}
$$

where X_t denotes the log forward process and A_t the variance. We have an analytical expression for its ChF.

Example: Heston Bermudan

We can also price Bermudan and discretely monitored barrier options.

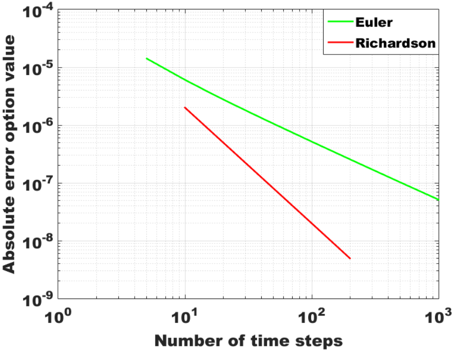

Figure: Bermudan put option with $s = 2$, $a = 0.2$, $\nu = 0.3$, $\rho = -0.2$, $T = 0.1$, strike $K = 1.9$, 5 early-exercise dates.

Solving BSDEs

We have a more general derivative with payoff $g(F_T)$ at time T, what is its value at time 0?

For example: the value of a European option under the $\mathbb{P}$-measure.

We create a self-financing portfolio Y, consisting of assets depending on F and α. This results in a BSDE:

$$\begin{aligned} \mathrm{d}Y_t &= -f(t, F_t, \alpha_t, Y_t, \mathbf{Z}_t)\,\mathrm{d}t + \left(Z_t^1 + \rho Z_t^2\right)\mathrm{d}W_t^1 + \sqrt{1-\rho^2}Z_t^2\,\mathrm{d}W_t^2, \\ Y_T &= g(F_T) \end{aligned}$$

We discretize this BSDE and use the COS method and the discretized FSDEs to approximate Y_0. There is a direct link between the processes Z_t^1 and Z_t^2 and the Δ-hedging strategy.

Geometric basket call option under $\mathbb{P}$-measure.

Both assets follow a geometric Brownian motion (GBM):

$$\begin{cases} \mathrm{d}S_t^1 = \mu_1 S_t^1 \,\mathrm{d}t + \sigma_1 S_t^1 \,\mathrm{d}W_t^1, & S_0^1 = s_1, \\ \mathrm{d}S_t^2 = \mu_2 S_t^2 \,\mathrm{d}t + \rho\sigma_2 S_t^2 \,\mathrm{d}W_t^1 + \sqrt{1-\rho^2}\sigma_2 S_t^2 \,\mathrm{d}W_t^2, & S_0^2 = s_2. \end{cases}$$

The payoff function is $g\left(S_T^1, S_T^2\right) = \max\left(\sqrt{S_T^1}\sqrt{S_T^2} - K, 0\right)$.

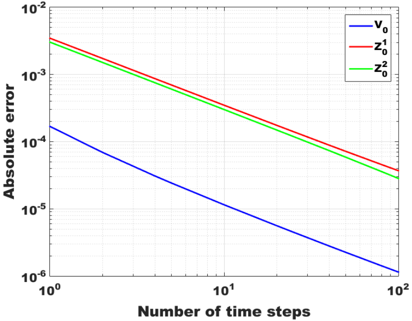

Figure: Basket option with $s_1 = 0.90,\ s_2 = 1.10,\ \mu_1 = 0.1,\ \mu_2 = 0.1,\ \sigma_1 = 0.2,\ \sigma_2 = 0.3,\ \rho = 0.25,\ r = 0.04,\ T = 0.1, K = 1$.

SABR call option under $\mathbb{P}$-measure.

The self-financing portfolio consists of stocks and assets depending on the volatility, for example a volatility swap.

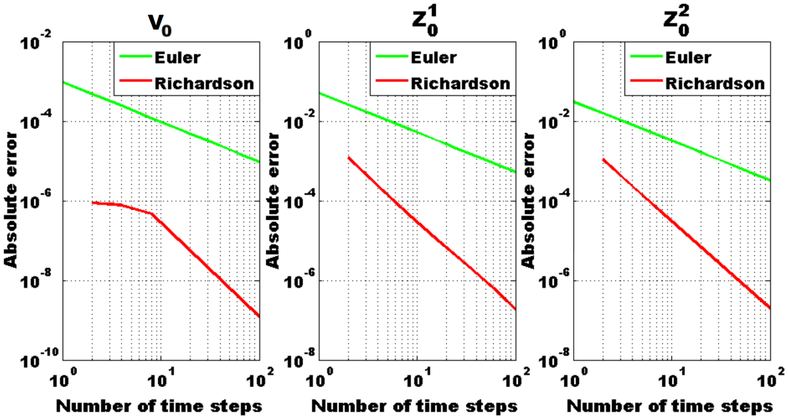

Figure: SABR call option with $f = 2,\ \alpha = 0.4,\ \mu = 0.2,\ \beta = 0.8,$ $\nu = 0.4,\ \rho = 0.25,\ r = 0.04,\ T = 0.1, K = 1.9.$

Summary

- We use the COS method for pricing derivatives, also when the ChF is not known (SABR model).
- We are able to price European options for multiple strikes at once under the SABR model.
- We apply Richardson extrapolation to achieve second order convergence with less computational costs.
- We use the COS method to solve BSDEs (BCOS method).